

$N = 2$ Superconformal Nets

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Abstract

We provide an Operator Algebraic approach to $N = 2$ chiral Conformal Field Theory and set up the Noncommutative Geometric framework. Compared to the $N = 1$ case, the structure here is much richer. There are naturally associated nets of spectral triples and the JLO cocycles separate the Ramond sectors. We construct the $N = 2$ superconformal nets of von Neumann algebras in general, classify them in the discrete series $c < 3$, and study spectral flow. We prove the coset identification for the $N = 2$ super-Virasoro nets with $c < 3$, a key result whose equivalent in the vertex algebra context is seemingly not complete. Finally, the chiral ring is discussed in terms of net representations.

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1 Introduction

Quantum Field Theory (QFT) describes a quantum system with infinitely many degrees of freedom and, from a geometrical viewpoint, can be regarded as an infinite-dimensional noncommutative manifold. It thus becomes a natural place for merging the classical infinite-dimensional calculus with the noncommutative quantum calculus. As explained in [41], a QFT index theorem should manifest itself in this setting and Noncommutative Geometry should provide a natural framework.

Within this program, localized representations with finite Jones index should play a role analogous to the one of elliptic operators in the classical framework. One example of this structure was suggested in the black hole entropy context, the Hamiltonian was regarded in analogy with the (infinite dimensional promotion of the) Laplacian and spectral analysis coefficients were indeed identified with index invariants for the net and its representations [36]. However, following Connes [12], the notion of spectral triple is the basis for a Dirac operator analogue and this naturally leads to exploring the supersymmetric context.

A particularly interesting context where to look for this setting is provided by chiral Conformal Field Theory in two spacetime dimensions (CFT), a building block for general 2D CFT. There are several reasons why CFT is suitable for our purposes. On the one hand, the Operator Algebraic approach to QFT has been particularly successful within the CFT frame leading to a deep, model independent description and understanding of the underlying structure. On the other hand, there are different, geometrically based approaches to CFT suggesting a Noncommutative Geometric interpretation ought to exist, and in which fields represent the noncommutative variables. Since the root of Connes' Noncommutative Geometry is Operator Algebraic, one is naturally led to explore its appearance within local conformal nets of von Neumann algebras.

A first step in this direction was taken in [9] with the construction and structure analysis of the $N = 1$ superconformal nets of von Neumann algebras, the prime class of nets combining conformal invariance and supersymmetry. Indeed a spectral triple, the basic underlying object in Noncommutative Geometry, generalizes and abstracts the notion of Dirac operator, so the natural QFT models to find it are the supersymmetric ones where the supercharge operator is a an odd square root of the Hamiltonian.

Indeed a net of spectral triples has been later constructed in [8], associated with Ramond representations of the $N = 1$ super-Virasoro net, the most elementary superconformal net of von Neumann algebras. In particular the irreducible, unitary positive energy representation of the Ramond algebra with central charge c and minimal lowest weight $h = c/24$ is graded and gives rise to a net of even θ -summable spectral triples with non-zero Fredholm index.

The $N = 1$ super-Virasoro algebra is an infinite dimensional Lie algebra generated by the Virasoro algebra and the Fourier modes of one Fermi field of conformal dimension $3/2$. There are higher level super-Virasoro algebras: the $N = 2$ one is generated by the Virasoro algebra and the Fourier modes of two Fermi fields and a $U(1)$ -current that generates rotations associated with the symmetry of the two Fermi fields. $N = 2$ superconformal nets will be extensions of a net associated with the $N = 2$ super-Virasoro net. One may continue the procedure even to $N = 4$, where four Fermi fields are present acted upon by $SU(2)$ -currents [38]. The various supersymmetries play crucial roles in several physical contexts, in particular in phase transitions of solid state physics and on the worldsheets of string theory.

This paper is devoted to the construction and analysis of the $N = 2$ superconformal nets. As is known, the passage from the $N = 1$ to the $N = 2$ case is not a matter of generalizing and extending results because a new and more interesting structure does appear by considering $N = 2$ superconformal models, although the definition of the respective nets is similar.

After summing up basic general preliminaries, we begin our analysis in Section 3, of course, constructing the $N = 2$ super-Virasoro net of von Neumann algebras by integrating the corresponding infinite-dimensional Lie algebra representation proving the necessary local energy bounds. The representations of the (vacuum) net will correspond to the representations of the $N = 2$ super-Virasoro algebra and will be of Neveu-Schwarz or Ramond type [2, 13], where Ramond representations are actually solitonic. This goes all in complete analogy to the $N = 1$ case [9].

At this point, however, there appears a remarkable new feature of the $N = 2$ super-Virasoro algebra: the appearance of the spectral flow, a “homotopy” equivalence between the Neveu-Schwarz and the Ramond algebra in the sense that there exists a deformation of one into the other. Solitonic Ramond representations of the nets are thus in correspondence with true (DHR) representations, an important fact of later use to us.

Before proceeding further, Section 5 is devoted to clarifying a key point of our paper: the identification for the $N = 2$ super-Virasoro nets with $c < 3$ as a coset for the inclusion $\mathcal{A}_{U(1)_{2n+4}} \subset \mathcal{A}_{SU(2)_n} \otimes \mathcal{A}_{U(1)_4}$. This identification is equivalent to the corresponding coset identification at the Lie algebra (or vertex algebra) level and it is moreover equivalent to the identification of the corresponding characters, cf. [34, 9] for the analogous statements in the $N = 0, 1$ cases. Accordingly, it is equivalent to the correctness of the known $N = 2$ character formulae for the discrete series representations, see e.g. [19, 17].

The $N = 2$ character formulae for the unitary representations with $c < 3$ were first derived (independently) by Dobrev [14], Kiritsis [39] and Matsuo [43]. Although these formulae appear to be universally accepted, a closer look to the literature seems to indicate that the mathematical validity of the proofs which have been proposed so far and of related issues of the representation theory of the $N = 2$ superconformal algebras is rather controversial, see [16, 19, 17, 15, 30] and [20, 21, 26, 27]. For this reason, we think that it is useful to give in this paper an independent complete mathematical proof of the $N = 2$ coset identification (and consequently of the $N = 2$ character formulae). Our proof, which we believe in any case to be of independent interest, is obtained largely through Operator Algebraic methods, a point that is certainly emblematic of the effectiveness and power of Operator Algebras.

We can then proceed with the classification of the $N = 2$ superconformal minimal models in Section 6, i.e., suitable extensions of the $N = 2$ super-Virasoro net. As in the local case [34], there are simple current series and exceptional nets. The proof is again based on combinatorics and subfactor methods.

The Noncommutative Geometric analysis starts in Section 7 where we construct the nets of spectral triples associated with Ramond representations of the $N = 2$ super-Virasoro algebra.

The main results in Noncommutative Geometry then are collected Section 8, where we consider the JLO cocycles for the spectral triples and pair them with K-theory. This pairing is nondegenerate and allows to separate, by means of certain characteristic projections, all Ramond vacuum sectors (the Ramond sectors with lowest weight $c/24$). Hence, for the first time Noncommutative Geometry is used to separate net representations, relying however also on the rich structure of the $N = 2$ superconformal context; in the $N = 1$ there was only one Ramond vacuum sector and the index pairing provides no insight there.

Our last Section 9 is dedicated to the study of the chiral ring (for the minimal models) from an operator algebraic point of view. The chiral ring is here defined and generated by the chiral primary sectors, a certain subset of Neveu-Schwarz sectors, and the monoidal product by means of truncated fusion rules, hence without direct reference to pointlike localized fields. However, the algebraic structure of the chiral ring coincides with the one provided by the operator product expansion of chiral primary fields. The spectral flow (at a specific value) is known to connect the chiral primary sectors with the vacuum Ramond sectors and we illustrate and exploit this in our setting, including some comments and hints for future work.

2 Preliminaries on superconformal nets

We provide here a very brief summary on graded-local conformal nets, just as much as we need to understand the general construction in the subsequent sections. The concept is an extension of local conformal nets and has been introduced and worked out in [9, Sect. 2,3,4]; we refer to that paper and [8] for a deeper study.

Let S^1 be the unit circle and let $\text{Diff}(S^1)$ be the infinite-dimensional (real) Lie group of orientation-preserving smooth diffeomorphisms of S^1 . The group $\text{PSL}(2, \mathbb{R})$ of Möbius transformations acts on S^1 and is a three-dimensional subgroup of $\text{Diff}(S^1)$. Let \mathcal{I} denote the set of nonempty and non-dense open intervals of S^1 . For any $I \in \mathcal{I}$, I' denotes the interior of $S^1 \setminus I$. Given $I \in \mathcal{I}$, the subgroup $\text{Diff}(S^1)_I$ of diffeomorphisms localized in I is defined as the stabilizer of I' in $\text{Diff}(S^1)$ namely the subgroup of $\text{Diff}(S^1)$ whose elements are the diffeomorphisms acting trivially on I' . Then, for any $n \in \mathbb{N} \cup \{\infty\}$, $\text{Diff}(S^1)_I^{(n)}$ denotes the connected component of the identity of the pre-image of $\text{Diff}(S^1)_I$ in $\text{Diff}(S^1)^{(n)}$ under the covering map. Then we write $\mathcal{I}^{(n)}$ for the set of intervals in $S^{1(n)}$ which map to an element in \mathcal{I} under the covering map. Moreover, we often identify \mathbb{R} with $S^1 \setminus \{-1\}$ by means of the Cayley transform, and we write $\mathcal{I}_{\mathbb{R}}$ (or $\tilde{\mathcal{I}}_{\mathbb{R}}$) for the set of bounded open intervals (and open half-lines, respectively) in \mathbb{R} .

Definition 2.1. A *graded-local conformal net* \mathcal{A} on S^1 is a map $I \mapsto \mathcal{A}(I)$ from the set of intervals \mathcal{I} to the set of von Neumann algebras acting on a common infinite-dimensional separable Hilbert space \mathcal{H} which satisfy the following properties:

- (A) *Isotony.* $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ if $I_1, I_2 \in \mathcal{I}$ and $I_1 \subset I_2$.
- (B) *Möbius covariance.* There is a strongly continuous unitary representation U of

$\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(\dot{g}I), \quad g \in \mathrm{PSL}(2, \mathbb{R})^{(\infty)}, I \in \mathcal{I}.$$

- (C) *Positive energy.* The conformal Hamiltonian L_0 (i.e., the self-adjoint generator of the restriction of the U to the lift to $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ of the one-parameter anti-clockwise rotation subgroup of $\mathrm{PSL}(2, \mathbb{R})$) is positive.
- (D) *Existence and uniqueness of the vacuum.* There exists a U -invariant vector $\Omega \in \mathcal{H}$ which is unique up to a phase and cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.
- (E) *Graded locality.* There exists a self-adjoint unitary Γ (the grading unitary) on \mathcal{H} satisfying $\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$ for all $I \in \mathcal{I}$ and $\Gamma\Omega = \Omega$ and such that

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*, \quad I \in \mathcal{I},$$

where

$$Z := \frac{\mathbf{1} - i\Gamma}{1 - i}.$$

- (F) *Diffeomorphism covariance.* There is a strongly continuous projective unitary representation of $\mathrm{Diff}(S^1)^{(\infty)}$, denoted again by U , extending the unitary representation of $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ and such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(\dot{g}I), \quad g \in \mathrm{Diff}(S^1)^{(\infty)}, I \in \mathcal{I},$$

and

$$U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I'), g \in \mathrm{Diff}(S^1)^{(\infty)}, I \in \mathcal{I}.$$

A *local conformal net* is a graded-local conformal net with trivial grading $\Gamma = \mathbf{1}$. The *even subnet* of a graded-local conformal net \mathcal{A} is defined as the fixed point subnet \mathcal{A}^γ , with grading gauge automorphism $\gamma = \mathrm{Ad} \Gamma$. It can be shown that the projective representation U of $\mathrm{Diff}(S^1)^\infty$ commutes with Γ , cf. [9, Lem.10]. Accordingly the restriction of \mathcal{A}^γ to the even subspace of \mathcal{H} is a local conformal net with respect to the restriction to this subspace of the projective representation U of $\mathrm{Diff}(S^1)^\infty$.

Some of the consequences [9, 24, 31, 10] of the preceding definition are:

- (1) *Reeh-Schlieder Property.* Ω is cyclic and separating for every $\mathcal{A}(I)$, $I \in \mathcal{I}$.
- (2) *Bisognano-Wichmann Property.* Let $I \in \mathcal{I}$ and let Δ_I, J_I be the modular operator and the modular conjugation of $(\mathcal{A}(I), \Omega)$. Then we have

$$U(\delta_I(-2\pi t)) = \Delta_I^{it}, \quad t \in \mathbb{R}.$$

Moreover unitary representation $U : \mathrm{PSL}(2, \mathbb{R})^{(\infty)} \mapsto B(\mathcal{H})$ extends to an (anti-)unitary representation of $\mathrm{PSL}(2, \mathbb{R}) \rtimes \mathbb{Z}/2$ determined by

$$U(r_I) = ZJ_I$$

and acting covariantly on \mathcal{A} . Here $(\delta_I(t))_{t \in \mathbb{R}}$ is (the lift to $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ of) the one-parameter dilation subgroup of $\mathrm{PSL}(2, \mathbb{R})$ with respect to I and r_I the reflection of the interval I onto the complement I' .

- (3) *Graded Haag Duality.* $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$, for $I \in \mathcal{I}$.

(4) *Outer regularity.*

$$\mathcal{A}(I_0) = \bigcap_{I \in \mathcal{I}, I \supset \bar{I}_0} \mathcal{A}(I), \quad I_0 \in \mathcal{I}.$$

(5) *Additivity.* If $I = \bigcup_{\alpha} I_{\alpha}$ with $I, I_{\alpha} \in \mathcal{I}$, then $\mathcal{A}(I) = \bigvee_{\alpha} \mathcal{A}(I_{\alpha})$.

(6) *Factoriality.* $\mathcal{A}(I)$ is a type III_1 -factor, for $I \in \mathcal{I}$.

(7) *Irreducibility.* $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$.

(8) *Vacuum Spin-Statistics theorem.* $e^{i2\pi L_0} = \Gamma$, in particular $e^{i2\pi L_0} = \mathbf{1}$ for local nets, where L_0 is the infinitesimal generator from above corresponding to rotations. Hence the representation U of $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ factors through a representation of $\mathrm{PSL}(2, \mathbb{R})^{(2)}$ ($\mathrm{PSL}(2, \mathbb{R})$ in the local case) and consequently its extension $\mathrm{Diff}(S^1)^{(\infty)}$ factors through a projective representation of $\mathrm{Diff}(S^1)^{(2)}$ ($\mathrm{Diff}(S^1)$ in the local case).

(9) *Uniqueness of Covariance.* For fixed Ω , the strongly continuous projective representation U of $\mathrm{Diff}(S^1)^{(\infty)}$ making the net covariant is unique.

In the sequel, G stands for either of the two groups $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{Diff}(S^1)$. From time to time we shall need *covering nets* of a given (graded-)local conformal net. By this we mean the following: a G -covariant net over $S^1^{(n)}$ is a family $(\mathcal{A}_n(I))_{I \in \mathcal{I}^{(n)}}$ such that

- $\mathcal{A}_n(I_1) \subset \mathcal{A}_n(I_2)$ if $I_1, I_2 \in \mathcal{I}^{(n)}$ and $I_1 \subset I_2$;
- there is a strongly continuous unitary representation U of $G^{(\infty)}$ on \mathcal{H} such that

$$U(g)\mathcal{A}_n(I)U(g)^* = \mathcal{A}_n(\dot{g}I), \quad g \in \mathrm{PSL}(2, \mathbb{R})^{(\infty)}, I \in \mathcal{I}^{(\infty)}.$$

The concept of representations is slightly more involved than in the ungraded case:

Definition 2.2. (1) A G -covariant soliton of a net \mathcal{A} over S^1 is a family $(\pi_I)_{I \in \bar{\mathcal{I}}_{\mathbb{R}}}$ of normal representations on a common Hilbert space \mathcal{H}_{π} with a projective unitary representation $U_{\pi} : G^{(\infty)} \rightarrow B(\mathcal{H}_{\pi})$ such that, for every $I \in \mathcal{I}_{\mathbb{R}}$ and $\mathbf{1}$ -neighbourhood $V_I \subset G^{(\infty)}$ with $V_I \cdot I \in \mathcal{I}_{\mathbb{R}}$:

$$U_{\pi}(g)\pi_I(x)U_{\pi}(g)^* = \pi_{\dot{g}I}(U(g)xU(g)^*), \quad g \in G^{(\infty)}, x \in \mathcal{A}(I).$$

(2) A G -covariant general soliton is a diffeomorphism-covariant soliton such that the restriction of \mathcal{A} to the even subnet \mathcal{A}^{γ} over S^1 is a DHR representation. In case $G = \mathrm{Diff}(S^1)$, we shall simply say *general soliton*. It is called a *(DHR) representation* if $\mathcal{I}_{\mathbb{R}}$ and $\bar{\mathcal{I}}_{\mathbb{R}}$ can be replaced by \mathcal{I} .

(3) A G -covariant general soliton π of a graded net (\mathcal{A}, γ) over S^1 is called *graded* if there exists a selfadjoint unitary $\Gamma_{\pi} \in B(\mathcal{H})$ such that

$$\Gamma_{\pi}\pi_I(x)\Gamma_{\pi} = \pi_I(\gamma(x)), \quad x \in \mathcal{A}(I), I \in \mathcal{I}_{\mathbb{R}}.$$

(4) A G -covariant graded general soliton π of a superconformal net \mathcal{A} is called *super-symmetric* if L_0^{π} admits an odd square-root.

Remark 2.3. It can be shown (using a straight-forward reasoning based on covariance relations) that a family $(\pi_I)_{I \in \mathcal{I}_{\mathbb{R}}}$ of normal representations of \mathcal{A} which is covariant with respect to a given projective unitary representation of $G^{(\infty)}$ extends automatically from $\mathcal{I}_{\mathbb{R}}$ to $\bar{\mathcal{I}}_{\mathbb{R}}$, thus defines a G -covariant soliton. We shall make use of this (simplifying) fact when considering explicit $N = 2$ super-Virasoro nets.

For the more common case of a (ungraded) *local* net \mathcal{B} over S^1 (like the even subnet $\mathcal{B} = \mathcal{A}^\gamma$) we recall the following associated global algebras, inspired by [23], and adapted and described in the current setting in [8, Def.2.3]:

Definition 2.4. The *universal C^* -algebra* $C^*(\mathcal{B})$ of \mathcal{B} is determined by the following properties:

- for every $I \in \mathcal{I}$, there are unital embeddings $\iota_I : \mathcal{B}(I) \rightarrow C^*(\mathcal{B})$, such that $\iota_{I_1|_{\mathcal{B}(I_2)}} = \iota_{I_2}$ whenever $I_1 \subset I_2$, and all $\iota_I(\mathcal{B}(I))$ together generate $C^*(\mathcal{B})$ as a C^* -algebra;
- for every representation π of \mathcal{B} on some Hilbert space \mathcal{H}_π , there is a unique $*$ -representation $\tilde{\pi} : C^*(\mathcal{B}) \rightarrow B(\mathcal{H}_\pi)$ such that

$$\pi_I = \tilde{\pi} \circ \iota_I, \quad I \in \mathcal{I}.$$

It can be shown to be unique up to isomorphism. Let $(\tilde{\pi}_u, \mathcal{H}_u)$ be the *universal representation* of $C^*(\mathcal{B})$: the direct sum of all GNS representations $\tilde{\pi}$ of $C^*(\mathcal{B})$. Since it is faithful, $C^*(\mathcal{B})$ can be identified with $\tilde{\pi}_u(C^*(\mathcal{B}))$. We call the weak closure $\text{vN}(\mathcal{B}) = \tilde{\pi}_u(C^*(\mathcal{B}))''$ the *universal von Neumann algebra* of \mathcal{B} . We shall drop the $\tilde{\cdot}$ sign henceforth.

Coming back to the general case of graded-local nets, a fundamental first consequence of Definition 2.2 is

Proposition 2.5 ([9, Sec.4.3]). *Let \mathcal{A} be a graded-local conformal net and π an irreducible G -covariant general soliton of \mathcal{A} . Then the following three conditions are equivalent:*

- π is graded,
- $\pi|_{\mathcal{A}^\gamma}$ is reducible,
- $\pi|_{\mathcal{A}^\gamma} \simeq \pi_+ \oplus \pi_+ \alpha =: \pi_+ \oplus \pi_-$,

with π_+ some irreducible localised DHR representation of \mathcal{A}^γ and α a localised DHR automorphism of \mathcal{A}^γ dual to the grading.

It can be shown that for irreducible graded π and under the assumption of finite statistical dimension on π_+ , we have the two possibilities

$$e^{i2\pi L_0^\pi} = e^{i2\pi L_0^{\pi_+}} \oplus \pm e^{i2\pi L_0^{\pi_+}} = e^{2\pi i h} \mathbf{1} \oplus \pm e^{2\pi i h} \mathbf{1},$$

so $e^{i4\pi L_0^\pi} = e^{4\pi i h} \mathbf{1}$ is a scalar, while in the irreducible ungraded case this is always trivially true. Here “+” will correspond to (R) in the following theorem, “−” to (NS) . Thus every irreducible general soliton of finite statistical dimension factorises through a representation of a net over $S^{1(2)}$, and it actually suffices now to consider graded-local conformal nets over $S^{1(n)}$ with $n = 1, 2$ and not higher – that is what we shall do henceforth.

Lemma 2.6. *If a G -covariant soliton ρ on a graded-local net \mathcal{A} is such that $e^{2\pi i L_0^\rho}$ is either the identity operator or implements the grading, then $\rho|_{\mathcal{A}^\gamma}$ is a DHR representation, i.e., ρ is a general G -covariant soliton of \mathcal{A} .*

Proof. Let U_ρ be the covariance unitary representation of ρ . We can extend ρ to a representation of the promotion $\mathcal{A}^{(\infty)}$ to the universal cover $S^{1(\infty)}$ by setting $\rho_{gI} := \text{Ad}U_\rho(g) \cdot \rho_I$ for every $I \in \mathcal{I}^{(\infty)}$. As $U_\rho(4\pi) = e^{4\pi i L_0^\rho} = \mathbf{1}$, ρ defines actually a G -covariant representation of the double cover net $\mathcal{A}^{(2)}$ over $S^{1(2)}$. By assumption, $U_\rho(2\pi)$ commutes with the image of the restriction of ρ to the even subnet \mathcal{A}^γ of \mathcal{A} , so ρ is a DHR representation of \mathcal{A}^γ , cf. [9, Prop.19]. \square

Theorem 2.7 (cf. [9, Sec.4.3]). *Let \mathcal{A} be a graded-local conformal net over S^1 and let π be an irreducible G -covariant general soliton of \mathcal{A} such that $e^{i4\pi L_0^\pi}$ is a scalar, and denote $\pi|_{\mathcal{A}^\gamma} =: \pi_+ \oplus \pi_+ \alpha$ or $\pi|_{\mathcal{A}^\gamma} =: \pi_+$ with an irreducible representation π_+ of \mathcal{A}^γ (for graded or ungraded π , respectively). Then π is of either of the subsequent two types:*

(NS) π is actually a representation of \mathcal{A} ; equivalently,
 $e^{i2\pi L_0^\pi} = \Gamma_\pi$ implements the grading.

(R) π is not a representation but only a general soliton of \mathcal{A} ; equivalently,
 $e^{i2\pi L_0^\pi}$ is a scalar, hence does not implement the grading.

In case (NS), π is called a Neveu-Schwarz representation of \mathcal{A} , and in case (R), a Ramond representation, the latter being however actually only a general soliton, i.e., a representation of $\mathcal{A}^{(2)}$ over $S^{1(2)}$, and not a proper representation of \mathcal{A} . A direct sum of irreducible Neveu-Schwarz (Ramond) representations is again called a Neveu-Schwarz (Ramond) representation.

Locally, i.e., for a local representation of $G^{(2)}$ and in restriction to a net over \mathbb{R} , Neveu-Schwarz and Ramond representations of $\mathcal{A}_\mathbb{R}$ are equivalent. The difference arises when passing to the global setting, i.e., to a (global) representation of $G^{(2)}$ and the whole net over $S^{1(2)}$. In this context, we recall from [8, Def. 2.6] that a general soliton is called *supersymmetric* if L_0^π has an odd square-root, and [8, Prop. 2.10] then says:

Proposition 2.8. *An irreducible general soliton of a superconformal net \mathcal{A} is supersymmetric iff it is a Ramond representation.*

3 The $N = 2$ super-Virasoro net and its representations

Definition 3.1. For any $t \in \mathbb{R}$, the $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2,t}$ is the infinite-dimensional Lie superalgebra generated by linearly independent even elements L_n, J_n and odd elements G_r^\pm , where $n \in \mathbb{Z}$, $r \in \frac{1}{2} \mp t + \mathbb{Z}$, together with an even central element \hat{c} and with (anti-) commutation relations

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{\hat{c}}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right)G_{m+r}^\pm, \\ [G_r^+, G_s^-] &= 2L_{r+s} + (r - s)J_{r+s} + \frac{\hat{c}}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\ [G_r^+, G_s^+] &= [G_r^-, G_s^-] = 0, \\ [L_m, J_n] &= -nJ_{m+n}, \\ [G_r^\pm, J_n] &= \mp G_{r+n}^\pm, \\ [J_m, J_n] &= \frac{\hat{c}}{3}m\delta_{m+n,0}. \end{aligned}$$

The *Neveu-Schwarz (NS) $N = 2$ super-Virasoro algebra* is the super-Virasoro algebra with $t = 0$, while the *Ramond (R) $N = 2$ super-Virasoro algebra* is the one with $t = 1/2$. Sometimes we shall write simply $\text{SVir}^{N=2}$ for the Neveu-Schwarz $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2,0}$.

We remark that in the special case $t \in \frac{1}{2}\mathbb{Z}$, instead of G_r^\pm , one sometimes considers the modes

$$G_r^1 := \frac{G_r^+ + G_r^-}{\sqrt{2}}, \quad G_r^2 := -i \frac{(G_r^+ - G_r^-)}{\sqrt{2}},$$

but we shall not use them. For all $t \in \mathbb{R}$, the Lie superalgebra $\text{SVir}^{N=2,t}$ is equipped with a natural anti-linear involution, such that the adjoints of L_n , J_n , G_r^i are respectively L_{-n} , J_{-n} , G_{-r}^i , $i = 1, 2$ and \hat{c} is selfadjoint.

We are interested in linear vector space representations. These representations should satisfy the usual conditions explained in [7, Sect.4], in short, they should be unitary with respect to a suitable scalar product turning the vector space into a pre-Hilbert space, \hat{c} should be represented by a positive real scalar c , and L_0 should be diagonalisable with every eigenspace finite-dimensional and only positive eigenvalues. Note that in the NS case the positivity of L_0 follows automatically from the commutation relations $2L_0 = [G_{1/2}^i, G_{-1/2}^i]$, $i = 1, 2$. In the R case we have $2L_0 - c/12 = [G_0^+, G_0^-] \geq 0$ and hence L_0 is bounded from below. It then follows by unitarity that $c \geq 0$ ¹ and $L_0 \geq c/24 \geq 0$. Accordingly an irreducible unitary representation is completely determined by the corresponding irreducible unitary representation of the zero modes on the lowest energy subspace (the subspace of highest weight vectors). In the NS case the algebra of zero modes is abelian and irreducibility implies that the lowest energy subspace is one-dimensional and spanned by a single vector $\Omega_{c,h,q}$ of norm one such that $L_0\Omega_{c,h,q} = h\Omega_{c,h,q}$ and $J_0\Omega_{c,h,q} = q\Omega_{c,h,q}$. The real numbers c, h, q completely determine the representation (up to unitary equivalence). In the R case the algebra of zero modes is non-abelian and there are two possibilities. If the lowest energy h is equal to $c/24$, the lowest energy subspace must be one dimensional again spanned by a normalized common eigenvector $\Omega_{c,h,q}$ of L_0 and J_0 with eigenvalues h and q respectively and satisfying $G_0^+\Omega_{c,h,q} = G_0^-\Omega_{c,h,q} = 0$. In contrast if $h > c/24$ the lowest energy subspace must be two-dimensional. Then one can choose a common normalized eigenvector $\Omega_{c,h,q}^-$ for L_0 and J_0 , with eigenvalues h and q respectively by imposing the supplementary condition $G_0^+\Omega_{c,h,q}^- = 0$. The lowest energy subspace is spanned by $\Omega_{c,h,q}^-$ and $\Omega_{c,h,q-1}^+$ where $\Omega_{c,h,q-1}^+ = (2h - c/12)^{-\frac{1}{2}}G_0^-\Omega_{c,h,q}^-$ is normalized and satisfies $L_0\Omega_{c,h,q-1}^+ = h\Omega_{c,h,q-1}^+$ and $J_0\Omega_{c,h,q-1}^+ = (q-1)\Omega_{c,h,q-1}^+$. With the above convention the numbers c, h, q completely determine the representation also in the R case. As in the cases $N = 0, 1$, unitarity gives restrictions on the possible values of c, h, q . The situation is described in

Theorem 3.2 ([2], [13], [32]). *For any irreducible unitary representation of the Neveu-Schwarz $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2,0}$ the corresponding values of c, h, q satisfy one of the following conditions:*

$$\text{NS1 } c \geq 3 \text{ and } 2h - 2nq + \left(\frac{c}{3} - 1\right)(n^2 - \frac{1}{4}) \geq 0 \text{ for all } n \in \frac{1}{2} + \mathbb{Z}.$$

$$\begin{aligned} \text{NS2 } c \geq 3 \text{ and } 2h - 2nq + \left(\frac{c}{3} - 1\right)(n^2 - \frac{1}{4}) &= 0, \\ 2h - 2(n + \text{sgn}(n))q + \left(\frac{c}{3} - 1\right)\left[(n + \text{sgn}(n))^2 - \frac{1}{4}\right] &< 0 \text{ for some } n \in \frac{1}{2} + \mathbb{Z} \text{ and} \\ 2\left(\frac{c}{3} - 1\right)h + \frac{1}{4}\left(\frac{c}{3} + 1\right)^2 - q^2 - \frac{1}{4}\left(\frac{c}{3} - 1\right)^2 &\geq 0. \end{aligned}$$

$$\begin{aligned} \text{NS3 } c &= \frac{3n}{n+2}, \quad h = \frac{l(l+2)-m^2}{4(n+2)}, \quad q = -\frac{m}{n+2}, \text{ where } n, l, m \in \mathbb{Z} \text{ satisfy } n \geq 0, 0 \leq l \leq n, \\ l + m &\in 2\mathbb{Z} \text{ and } |m| \leq l. \end{aligned}$$

For any irreducible unitary representation of the Ramond $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2,\frac{1}{2}}$ the corresponding values c, h, q satisfy one of the following conditions:

¹If ψ is an eigenvector of L_0 then $0 \leq (L_{-n}\psi, L_{-n}\psi) = (\psi, [L_n, L_{-n}]\psi) = 2n(\psi, L_0\psi) + \frac{c}{12}(n^3 - n)$ for all sufficiently large positive integers n . Hence c must be a non-negative real number.

R1 $c \geq 3$ and $2h - 2n(q - \frac{1}{2}) + (\frac{c}{3} - 1)(n^2 - \frac{1}{4}) - \frac{1}{4} \geq 0$ for all $n \in \mathbb{Z}$

R2 $c \geq 3$ and $2h - 2n(q - \frac{1}{2}) + (\frac{c}{3} - 1)(n^2 - \frac{1}{4}) - \frac{1}{4} = 0$,
 $2h - 2(n + \operatorname{sgn}(n - \frac{1}{2}))(q - \frac{1}{2}) + (\frac{c}{3} - 1) \left[(n + \operatorname{sgn}(n - \frac{1}{2}))^2 - \frac{1}{4} \right] - \frac{1}{4} < 0$ for some
 $n \in \mathbb{Z}$ and $2(\frac{c}{3} - 1)(h - \frac{c}{24}) + \frac{1}{4}(\frac{c}{3} + 1)^2 - (q - \frac{1}{2})^2 - \frac{1}{4}(\frac{c}{3} - 1)^2 \geq 0$.

R3 $c = \frac{3n}{n+2}$, $h = \frac{l(l+2)-m^2}{4(n+2)} + \frac{1}{8}$, $q = -\frac{m}{n+2} + \frac{1}{2}$, where $n, l, m \in \mathbb{Z}$ satisfy $n \geq 0$,
 $0 \leq l \leq n$, $l + m + 1 \in 2\mathbb{Z}$ and $|m - 1| \leq l$.

Conditions NS1, NS3, R1, R3, are also sufficient, namely if values c, h, q satisfy one of them then there exists a corresponding irreducible unitary representation. In particular all the values in the discrete series of representations (conditions NS3 and R3) with $c = 3n/(n+2)$ are realized by the coset construction for the inclusion $U(1)_{2n+4} \subset SU(2)_n \otimes \text{CAR}^{\otimes 2}$ for every nonnegative integer n .

For every allowed value of c , there is a corresponding unique representation of the Neveu-Schwarz $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2,0}$ with $h = q = 0$, the **vacuum representation**. Moreover, if $n \in 2 + 4\mathbb{Z}$, then there are Ramond representations with $h = c/24$ (and several values for q) and we call them **vacuum Ramond representations**. Notice from Definition 3.1 that in contrast to the case $N = 1$, every representation π is automatically graded by $e^{i\pi J_0^\pi}$: clearly, this element commutes with all L_n^π and J_n^π , while it follows (using e.g. (7.1) below with $s = \pi$) that it anticommutes with all $G_r^{\pm, \pi}$.

Our goal in this section is to define a net associated to the $N = 2$ super-Virasoro algebra. This will be done in the standard way by using certain unbounded selfadjoint fields (in the vacuum representation). Let π be an irreducible unitary positive energy lowest weight representation of $\text{SVir}_c^{N=2}$ as above with certain (c, h, q) . Then we denote the generators in that representation by $L_n^\pi, G_r^{i, \pi}, J_n^\pi$, and c^π (or simply c) will be one of the positive admissible numbers. If π is the vacuum representation, we shall drop the superscript π .

We shall need the following linear energy bounds [4] and [9]: in every unitary representation π , we have

$$\begin{aligned} \|L_m^\pi \psi\| &\leq M_c(1 + |m|^{\frac{3}{2}})\|(\mathbf{1} + L_0^\pi)\psi\|, \\ \|G_r^{i, \pi} \psi\| &\leq (2 + \frac{c}{3}r^2)^{\frac{1}{2}}\|(\mathbf{1} + L_0^\pi)^{\frac{1}{2}}\psi\|, \\ \|J_m^\pi \psi\| &\leq (1 + c|m|)^{1/2}\|(\mathbf{1} + L_0^\pi)^{\frac{1}{2}}\psi\|, \quad \psi \in C^\infty(L_0^\pi) \end{aligned} \tag{3.1}$$

with a constant $M_c > 0$ depending only on c .

For smooth and localised functions $f \in C_I^\infty(S^1)$ with $I \in \mathcal{I}_{\mathbb{R}}$, the Fourier coefficients f_n are rapidly decreasing, and owing to the above linear energy bounds, the formal sums

$$\sum_{n \in \mathbb{Z}} f_n L_n^\pi, \quad \sum_{r \in \mp t + \frac{1}{2} + \mathbb{Z}} f_r G_r^{i, \pi}, \quad \sum_{n \in \mathbb{Z}} f_n J_n^\pi, \quad (i = 1, 2),$$

are densely defined closable operators with common invariant core $C^\infty(L_0)$ by a standard reasoning based on [18, Th.3.2] (for details consider e.g. [4] or [5, Sect.2.2]). We denote their selfadjoint closures, the so-called smeared fields, by $L^\pi(f)$, and analogously for $J^\pi(f)$ and $G^{\pm, \pi}(f)$ with $i = 1, 2$, respectively. The former two are actually selfadjoint. Concerning the latter one, we define $G^{i, \pi}(f)$ as the closures of the essentially selfadjoint smeared fields

$$\frac{1}{\sqrt{2}}(G^{+, \pi}(f) + G^{-, \pi}(f))|_{C^\infty(L_0)}, \quad \frac{1}{i\sqrt{2}}(G^{+, \pi}(f) - G^{-, \pi}(f))|_{C^\infty(L_0)}.$$

Since

$$[X^\pi(f), Y^\pi(g)] = 0, \quad \text{supp}(f) \cap \text{supp}(g) = \emptyset,$$

for X and Y any of the fields L, G^i, J , we can define a graded local net of von Neumann algebras by

$$\mathcal{A}_\pi(I) = \{e^{iJ^\pi(f)}, e^{iL^\pi(f)}, e^{iG^{\pi, (i)}(f)} : f \in C^\infty(S^1), \text{supp } f \subset I\}'', \quad I \in \mathcal{I}_\mathbb{R}. \quad (3.2)$$

If π is the vacuum representation with central charge c , we shall denote by \mathcal{A}_c the corresponding net on \mathbb{R} and call it the $N = 2$ super-*Virasoro* net on \mathbb{R} with central charge c . The extension to S^1 whose existence is guaranteed by the following theorem will be denoted again by \mathcal{A}_c and called the $N = 2$ *super- Virasoro net* with central charge c .

Theorem 3.3. *The family $(\mathcal{A}_c(I))_{I \in \mathcal{I}_\mathbb{R}}$ extends to a diffeomorphism covariant graded-local conformal net $\mathcal{A}_c = (\mathcal{A}_c(I))_{I \in \mathcal{I}}$ over S^1 .*

Proof. The proof goes in complete analogy to the one of [9, Th.33] where $N = 1$. We briefly recall that first, we have to construct a representation of $\text{Diff}(S^1)^{(2)}$ such that all J, G^i, L transform covariantly with respect to the restriction of that representation to $\text{PSL}(2, \mathbb{R})^{(2)}$. It is obtained as the integration of the given representation of the Lie algebra generated by L_{-1}, L_0, L_1 (the complexification of Lie algebra of $\text{PSL}(2, \mathbb{R})$). In fact, going through all the steps there, we just have to notice that (in the notation used there)

$$\frac{d}{dt} (J(\beta_0(e^{tf_1})f_2)\psi_0)|_{t=0} = [L(f_1), J(f_2)]\psi_0,$$

where $\psi_0 \in C^\infty(L_0)$, $f_1, f_2 \in C^\infty(S^1)$, and

$$\beta_0(g)f_i(z) := f_i(g^{-1}(z)), \quad g \in \text{PSL}(2, \mathbb{R})^{(2)}, z \in S^1.$$

Once this is done, we show cyclicity of the vacuum vector Ω for the net, and finally extension of $\text{PSL}(2, \mathbb{R})$ to diffeomorphism covariance, literally as in [9, Th.33]. \square

Now that the net \mathcal{A}_c is defined, we would like to study its representations. The result follows immediately from the classification of the Lie algebra representations in Theorem 3.2: from the branching rules and the associated expressions for the characters in [37], we obtain a general soliton of the net for every representation in Theorem 3.2. Finally, in Section 7 we shall use

Theorem 3.4 ([11]). *Let $(\pi_I)_{I \in \mathcal{I}}$ be an irreducible general soliton of \mathcal{A}_c with lowest energy $h \geq 0$ and charge $q \in \mathbb{R}$. Then there is a unitary irreducible highest weight representation π of the N -super-*Virasoro* algebra with lowest weight h and charge q such that*

$$\pi_I(e^{iL(f)}) = e^{iL^\pi(f)}, \quad \pi_I(e^{iG^i(f)}) = e^{iG^{i, \pi}(f)}, \quad \pi_I(e^{iJ(f)}) = e^{iJ^\pi(f)},$$

for $f \in C^\infty(S^1)_I$, $i = 1, 2$, $I \in \mathcal{I}_\mathbb{R}$. Moreover, $(\pi_I)_{I \in \mathcal{I}^{(2)}}$ is a Neveu-Schwarz (Ramond) representation in the sense of Theorem 2.7 iff π is a Neveu-Schwarz (Ramond) representation in the sense of Theorem 3.2. In particular, there is a one-to-one correspondence between general solitons of the net and representations of the Lie superalgebra.

Finally, a crucial fact owing to Fewster and Hollands [22, Th.4.1] is the following one: for every non-negative function $f \in C^\infty(S^1)$, the selfadjoint operator $L(f)$ is bounded from below.

4 Spectral flow for the $N = 2$ super-Virasoro net

A remarkable property of the $N = 2$ -super-Virasoro algebra is the “homotopic” equivalence of its Neveu-Schwarz and its Ramond algebra in the sense that there exists a deformation of one into the other, first discussed in [44]:

Definition 4.1. The *spectral flow* of the Lie algebra $\text{SVir}^{N=2}$ is the family of linear maps $\eta_t : \text{SVir}^{N=2,t} \rightarrow \text{SVir}^{N=2,0}$, with $t \in \mathbb{R}$, defined on the generators by

$$\begin{aligned}\eta_t(L_n) &:= L_n + tJ_n + \frac{c}{6}t^2\delta_{n,0}, \quad n \in \mathbb{Z} \\ \eta_t(J_n) &:= J_n + \frac{c}{3}t\delta_{n,0}, \quad n \in \mathbb{Z} \\ \eta_t(G_r^\pm) &:= G_{r \pm t}^\pm, \quad r \in \mp t + \frac{1}{2} + \mathbb{Z} \\ \eta_t(\hat{c}) &:= \hat{c}.\end{aligned}$$

In other words, the map η_t embeds $\text{SVir}^{N=2,t}$ into $\text{SVir}^{N=2}$.

Proposition 4.2. *The linear maps η_t are Lie superalgebra isomorphisms, so the Lie superalgebras $\text{SVir}^{N=2,t}$, $t \in \mathbb{R}$, are all isomorphic. In particular, the Neveu-Schwarz algebra and the Ramond algebra are isomorphic.*

Proof. Define a linear map $\eta'_t : \text{SVir}^{N=2,0} \rightarrow \text{SVir}^{N=2,t}$, given on the generators by

$$\begin{aligned}\eta'_t(L_n) &:= L_n - tJ_n + \frac{c}{6}t^2\delta_{n,0}, \quad n \in \mathbb{Z} \\ \eta'_t(J_n) &:= J_n - \frac{c}{3}t\delta_{n,0}, \quad n \in \mathbb{Z} \\ \eta'_t(G_r^\pm) &:= G_{r \mp t}^\pm, \quad r \in \frac{1}{2} + \mathbb{Z} \\ \eta'_t(\hat{c}) &:= \hat{c}.\end{aligned}$$

It is straightforward to check that it is an inverse for η_t , so we have bijectivity. Moreover, using Definition 3.1, one finds that η_t is a Lie algebra homomorphism, so we are done. \square

In order to treat the local algebras, we need the spectral flow on smeared fields. For any $t \in \mathbb{R}$ and $f \in C^\infty(S^1)_I$ with $I \in \mathcal{I}_\mathbb{R}$, the modified Fourier coefficients

$$f_r := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ir\theta} f(e^{i\theta}) d\theta, \quad r \in t + \frac{1}{2} + \mathbb{Z}, \quad (4.1)$$

are rapidly decreasing.

Consider now $\text{SVir}^{N=2,t}$ and denote by π_0 the vacuum representation of the Lie superalgebra $\text{SVir}^{N=2}$ with a fixed central charge c , the unique representation with lowest weight $h = 0$. With $L_n^t := \pi_0(\eta_t(L_n))$, $J_n^t := \pi_0(\eta_t(J_n))$, $G_r^{\pm,t} := \pi_0(\eta_t(G_r^\pm))$ (we suppress the superscript \cdot^t when $t = 0$), we may consider the formal power series

$$\sum_{n \in \mathbb{Z}} f_n J_n^t, \quad \sum_{r \in \mp t + \frac{1}{2} + \mathbb{Z}} f_r G_r^{\pm,t}, \quad \sum_{n \in \mathbb{Z}} f_n L_n^t, \quad (4.2)$$

on \mathcal{H}_0 , which are densely defined and closable with core $C^\infty(L_0)$, and we denote their closures by $J^t(f), G^{\pm,t}(f), L^t(f)$. In the light of the analogous definition with $t = 0$, we also define the selfadjoint fields

$$G^{1,t}(f) := \frac{\left((G^{+,t}(f) + G^{-,t}(f))|_{C^\infty(L_0)}\right)^-}{\sqrt{2}}, \quad G^{2,t}(f) := \frac{\left((G^{+,t}(f) - G^{-,t}(f))|_{C^\infty(L_0)}\right)^-}{i\sqrt{2}}, \quad (4.3)$$

and $J^t(f), G^{\pm,t}(f), L^t(f)$ are then selfadjoint on \mathcal{H} and affiliated with $\mathcal{A}_c(I)$ if $\text{supp } f \subset I$.

We shall see that the action of η_t on the even generators corresponds to a $U(1)$ -automorphism ρ_q [3, Sect.2] with charge $q = tc/3$, and our particular interest lies in the case $t \in \frac{1}{2}\mathbb{Z}$.

Theorem 4.3. *For every $t \in \mathbb{R}$, there is a $\text{PSL}(2, \mathbb{R})$ -covariant general soliton $\bar{\eta}_t$ of \mathcal{A}_c over $S^1 \setminus \{-1\} \simeq \mathbb{R}$ such that $\bar{\eta}_{t,I}$, $I \in \mathcal{I}_{\mathbb{R}}$, is the automorphism given by*

$$\bar{\eta}_{t,I}(e^{iX(f)}) = e^{iX^t(f)}, \quad f \in C^\infty(S^1 \setminus \{-1\})_I, \quad X = J, G^1, G^2, L.$$

$\bar{\eta}_t$ is equivalent to the α^\pm -induction of a localised $U(1)$ -current endomorphism ρ_q with charge $q = \frac{c}{3}t$. For $t \in \mathbb{Z}$, $\bar{\eta}_t$ extends to a Neveu-Schwarz representation of the graded-local net \mathcal{A}_c , while for $t \in \frac{1}{2} + \mathbb{Z}$, $\bar{\eta}_t$ extends to a Ramond representation of \mathcal{A}_c .

The proof proceeds in several steps. First, let us define representations for each local algebra. Let $I \in \mathcal{I}_{\mathbb{R}}$ and let $\phi_{t,I} \in C_c^\infty(S^1 \setminus \{-1\})$ be such that $\phi_{t,I}|_I = \frac{t}{i} \log$ where \log is normalised such that $\log(1) = 0$. Set

$$\bar{\eta}_{t,I}(x) := \text{Ad}(e^{iJ(\phi_{t,I})})(x), \quad x \in \mathcal{A}_c(I). \quad (4.4)$$

Lemma 4.4. *$X(f) \mapsto X^t(f)$, for $f \in C^\infty(S^1)_I$, is a well-defined linear map, implemented by $e^{iJ(\phi_{t,I})}$. It integrates to $\bar{\eta}_{t,I}$ on $\mathcal{A}_c(I)$.*

Proof. Let $f \in C_I^\infty(S^1)$. Then by the commutation relations obtained from smearing the relations in Definition 4.1, we have

$$J^t(f) = J(f) + q \int_I f = \text{Ad}(e^{iJ(\phi_{t,I})})(J(f))$$

and

$$L^t(f) = L(f) + t \int_I f J(f) + (t \int_I f)^2 = \text{Ad}(e^{iJ(\phi_{t,I})})(L(f)).$$

On the fermionic generators we obtain

$$G^{t,\pm}(f) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r G_r^{t,\pm} = \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_{r \mp t} G_r^\pm = G^\pm(e^{\pm i t \phi} f),$$

where $\iota_t : e^{i\theta} \mapsto e^{i t \theta}$, with θ in a suitable neighbourhood of 0 depending on t . We claim that

$$G^\pm(e^{\pm i t \phi} f) = e^{i t J(\phi)} G^\pm(f) e^{-i t J(\phi)}, \quad (4.5)$$

with $\phi = \phi_{1,I}$ here. Then since the RHS of these equations is given by the action of $\text{Ad}(e^{i t J(\phi)})$ on the smeared fields, we obtain a well-defined homomorphism η_t . It clearly integrates to a normal homomorphism of the generated von Neumann algebra $\mathcal{A}_c(I)$, namely $\bar{\eta}_{t,I}$.

The idea of the proof of (4.5) is similar to the one of Theorem 3.3. Let U be the strongly continuous unitary irreducible representation of the $\mathbf{1}$ -connected component of

the loop group $\mathrm{L} \mathrm{U}(1)$ (instead of $\mathrm{Diff}(S^1)$ there) on the vacuum representation space \mathcal{H} obtained by integrating the vacuum representation π_0 of the $\mathfrak{u}(1)$ -current algebra as in [29] or [24], i.e.,

$$U(e^{i\phi}) = e^{iJ(\phi)} \in B(\mathcal{H}), \quad \phi \in C^\infty(S^1, S^1) = \mathrm{L} \mathfrak{u}(1).$$

Then $U(e^{i\phi})$ preserves the core $C^\infty(L_0)$ because all $e^{itJ(\phi)}$ do so [45, Sec.2], and, for every $\psi_0 \in C^\infty(L_0)$, the map

$$t \in \mathbb{R} \mapsto U(e^{it\phi})\psi_0 \in C^\infty(L_0)$$

is continuously differentiable. Let $\psi_0 \in C^\infty(L_0)$, recall that $e^{itJ(\phi)}$ preserves $C^\infty(L_0)$, and define

$$\psi^\pm(t) := G^\pm(e^{\pm it\phi} f) e^{itJ(\phi)} \psi_0, \quad t \in \mathbb{R}.$$

Then the above together with the linear energy bounds for G yield the differentiability of $t \mapsto \psi^\pm(t)$ and the commutation relations in Definition 3.1 imply

$$\begin{aligned} \frac{d}{dt} \psi^\pm(t) &= \pm G^\pm(\phi e^{\pm it\phi} f) e^{itJ(\phi)} \psi_0 + G^\pm(e^{\pm it\phi} f) J(\phi) e^{itJ(\phi)} \psi_0 \\ &= J(\phi) G^\pm(e^{\pm it\phi} f) e^{itJ(\phi)} \psi_0 \\ &= J(\phi) \psi^\pm(t). \end{aligned}$$

The solution of this differential equation with given initial value is

$$\psi^\pm(t) = e^{itJ(\phi)} \psi^\pm(0) = e^{itJ(\phi)} G^\pm(f) \psi_0.$$

Since this holds true for every $\psi_0 \in C^\infty(L_0)$ and the latter is a core for all $G^\pm(f)$ and preserved by $e^{itJ(\phi)}$, we obtain

$$e^{itJ(\phi)} G^\pm(f) = G^\pm(e^{\pm it\phi} f) e^{itJ(\phi)},$$

hence (4.5). The latter one then implies, on $C^\infty(L_0)$,

$$\begin{aligned} G^{1,t}(f) &= \frac{1}{2}(G^{+,t}(f) + G^{-,t}(f)) = \frac{1}{2}(G^+(e^{it\phi} f) + G^-(e^{-it\phi} f)) \\ &= \frac{1}{2}(e^{itJ(\phi)} G^+(f) e^{-itJ(\phi)} + e^{itJ(\phi)} G^-(f) e^{-itJ(\phi)}) = e^{itJ(\phi)} G^1(f) e^{-itJ(\phi)} \end{aligned}$$

and analogously for $G^{2,t}(f)$. □

Lemma 4.5. *The family $(\bar{\eta}_{t,I})_{I \in \mathcal{I}_{\mathbb{R}}}$ forms a (locally normal) $\mathrm{PSL}(2, \mathbb{R})$ -covariant soliton of the restricted net $\mathcal{A}_c^{(0)}$.*

Proof. The local normality of $\bar{\eta}_{t,I}$ is obvious. Here we have to establish the covariance. Let U_q be the integration to $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ of the representation of the Lie algebra generated by L_{-1}^t, L_0^t, L_1^t , i.e., we have to define suitable $U_q : \mathrm{PSL}(2, \mathbb{R})^{(\infty)} \rightarrow B(\mathcal{H})$ such that

$$U_q(g) \bar{\eta}_{t,I}(x) U_q(g)^* = \bar{\eta}_{t,gI}(U(g)xU(g)^*), \quad x \in \mathcal{A}_c(I),$$

for $g \in \mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ close to $\mathbf{1}$ such that the closure of $I \cup gI$ is contained in a subinterval of $S^1 \setminus \{-1\}$. As before, $q = \frac{c}{3}t$. Then in the same way as in the proof of Theorem 3.3 we obtain a representation U_q of $\mathrm{PSL}(2, \mathbb{R})^{(\infty)}$ such that

$$U_q(g) X^t(f) U_q(g)^* = X^t(\beta_{n(X)}(g).f),$$

with β_n certain functions (the ones appearing in the proof of Theorem 3.3 and [9, Sec.6.3]) and subscripts $n(J) = 0$, $n(G^i) = 1/2$, $n(L) = 1$. Appealing to the preceding lemma, we find

$$\begin{aligned}
U_q(g)\bar{\eta}_{t,I}\left(e^{iX(f)}\right)U_q(g)^* &= U_q(g)\text{Ad}(e^{iJ(\phi_{t,I})})(e^{iX(f)})U_q(g)^* \\
&= U_q(g)e^{iX^t(f)}U_q(g)^* \\
&= e^{i(X^t(\beta_{n(X)}(g)\cdot f))} \\
&= \text{Ad}(e^{iJ(\phi_{t,\dot{g}I})})(e^{iX(\beta_{n(X)}(g)\cdot f)}) \\
&= \bar{\eta}_{t,\dot{g}I}\left(e^{iX(\beta_{n(X)}(g)\cdot f)}\right) \\
&= \bar{\eta}_{t,\dot{g}I}\left(U(g)e^{iX(f)}U(g)^*\right).
\end{aligned}$$

Thus, we have shown the $\text{PSL}(2, \mathbb{R})$ -covariance on the generators of $\mathcal{A}_c(I)$ with $I \in \mathcal{I}_{\mathbb{R}}$, hence for the whole net $\mathcal{A}_c^{(0)}$, so $\bar{\eta}_t$ becomes a $\text{PSL}(2, \mathbb{R})$ -covariant soliton in the sense of Definition 2.2. \square

Concerning covariance over S^1 or $S^{1(2)}$, an obvious problem in the above proof arises from the discontinuity in $g(-1)$ when considering the covariance of the odd fields. This discontinuity will vanish and the function will extend to a continuous function over S^1 or $S^{1(2)}$ iff $t \in \mathbb{Z}$ or $t \in \frac{1}{2} + \mathbb{Z}$, respectively, as we shall see in the proof of the main theorem, in particular (4.6). For the even fields, however this is no problem.

Lemma 4.6. *Given $t \in \mathbb{R}$, for every $I_0 \in \mathcal{I}_{\mathbb{R}}$, there is a unitary $u_{t,I_0} \in B(\mathcal{H})$ such that*

$$\bar{\eta}_{t,I} = \text{Ad}(u_{t,I_0}) \circ \alpha_{\rho_q^{I_0}, I}^+ = \text{Ad}(u_{t,I_0} e^{t\pi i J_0}) \circ \alpha_{\rho_q^{I_0}, I}^-, \quad I \in \mathcal{I}_{\mathbb{R}},$$

where $\rho_q^{I_0}$ is an endomorphism of the $U(1)$ -subnet of \mathcal{A}_c , with charge $q = \frac{c}{3}t$ and localised in J .

Proof. The global endomorphism of ρ_q of the $U(1)$ -subnet $\mathcal{A}_{U(1)} \subset \mathcal{A}_c$ is defined by

$$\rho_q(e^{iJ(f)}) = e^{i(J(f)+q \int f)}.$$

Given $I_0 \in \mathcal{I}_{\mathbb{R}}$, fix a smooth 2π -periodic function $h_{I_0} : \mathbb{R} \rightarrow \mathbb{R}$ which, restricted to $(-\pi, \pi)$, satisfies

$$h_{I_0}(\theta) = \begin{cases} \theta & : \theta < I_0 \\ \text{arbitrary} & : \theta \in I_0 \\ \theta - 2\pi & : \theta > I_0. \end{cases}$$

Then $\rho_q^{I_0} := \text{Ad}(e^{-tiJ(h_{I_0})}) \circ \rho_q$ is an endomorphism of $\mathcal{A}_{U(1)}$ localised in I_0 and equivalent to ρ_q . Recall [42] that the α -induced sectors of $\rho_q^{I_0}$ may be expressed as

$$\alpha_{\rho_q^{I_0}, I}^\pm = \text{Ad}(z(\rho_q^{I_0}, g_\pm)) = \text{Ad}(e^{tiJ(h_{g_\pm I_0} - h_{I_0})}),$$

where $g_\pm \in G$ are such that $g_- I_0 < I < g_+ I_0$.

For $x \in \mathcal{A}_c(I)$ we now obtain

$$\text{Ad}(e^{tiJ(h_{I_0})}) \circ \alpha_{\rho_q^{I_0}, I}^\pm(x) = \text{Ad}(e^{tiJ(h_{I_0})} e^{tiJ(h_{g_\pm I_0} - h_{I_0})})(x) = \text{Ad}(e^{tiJ(h_{g_\pm I_0})})(x).$$

Since $g_-I_0 < I < g_+I_0$, we see that $h_{g_+I_0}|_I = \iota|_I$, while $h_{g_-I_0}|_I = \iota|_I - 2\pi$. Thus the definition of $\phi_{t,I}$ and $\bar{\eta}_t$ finally implies

$$\text{Ad}(e^{tiJ(h_{I_0})}) \circ \alpha_{\rho_q^{J_0}, I}^\pm = \begin{cases} \bar{\eta}_{t,I} & : " + " \\ \text{Ad}(e^{-i2\pi tJ_0}) \circ \bar{\eta}_{t,I} & : " - " \end{cases},$$

so we are done, setting $u_{t,I_0} := \text{Ad}(e^{tiJ(h_{I_0})})$, independent of I . In particular, the α^\pm -induced sectors differ by the gauge automorphism $\text{Ad}(e^{i2\pi tJ_0})$ of the net \mathcal{A}_c , which is trivial if $t \in \mathbb{Z}$, i.e., when $\bar{\eta}_t$ becomes a Neveu-Schwarz endomorphism. \square

For either of the two cases Ramond and Neveu-Schwarz, i.e., for $t \in \frac{1}{2}\mathbb{Z}$, the restrictions of α_ρ^\pm to the even subnet \mathcal{A}_c^γ coincide since $e^{i\pi J_0}$ commutes with the latter. Thus $\bar{\eta}_t|_{\mathcal{A}_c^\gamma}$ extends in fact to \mathcal{I} , i.e., it is a DHR endomorphism, and $\bar{\eta}_t$ becomes a $\text{PSL}(2, \mathbb{R})^{(\infty)}$ -covariant general soliton. We remark that the general soliton is automatically diffeomorphism-covariant if \mathcal{A}_c is strongly additive [9, Prop.21].

Proof of Theorem 4.3. Lemma 4.5 tells us that $\bar{\eta}_t$ is a G -covariant soliton on the $\text{SVir}^{N=2}$ -net \mathcal{A}_c . Let us check that the assumptions of Lemma 4.4 are verified. As in the proof of Theorem 3.3, we obtain, for all possible r depending on the field $X = J, G^\pm, L$:

$$e^{isL_0^t} X_r^t e^{-isL_0^t} = e^{irs} X_r^t. \quad (4.6)$$

For $X^t = J^t, L^t$, we have $r \in \mathbb{Z}$, so the phase factor is $e^{2\pi ir} = 1$. For $X^t = G^{t,\pm}$ instead, we have $r \in \pm t + \frac{1}{2} + \mathbb{Z}$, so $e^{2\pi ir} = e^{\pm 2\pi i(t+1/2)}$. So $e^{2\pi iL_0^t}$ implements the grading (equals 1) precisely when $t \in \mathbb{Z}$ ($t \in \frac{1}{2} + \mathbb{Z}$, respectively).

Thus Lemma 2.6 implies that $\bar{\eta}_t$ is a $\text{PSL}(2, \mathbb{R})$ -covariant general soliton iff $t \in \frac{1}{2}\mathbb{Z}$, and it is either a Ramond or a Neveu-Schwarz $\text{PSL}(2, \mathbb{R})$ -covariant general soliton in the sense of Definition 2.2 and Theorem 2.7, depending on whether $t \in \frac{1}{2} + \mathbb{Z}$ or $t \in \mathbb{Z}$, respectively. The interpretation in terms of α -induction follows from Lemma 4.6.

We recall from Proposition 4.2 that precisely the values $t \in \mathbb{Z}$ ($t \in \frac{1}{2} + \mathbb{Z}$) correspond to the Neveu-Schwarz algebra (Ramond algebra, respectively), so we have a confirmation of Theorem 3.4 in the present setting: Neveu-Schwarz (Ramond) general solitons of the net correspond one-to-one to the Neveu-Schwarz (Ramond) representations of the Neveu-Schwarz (Ramond) super-Virasoro algebra.

Finally, it is clear from their definition that $(\bar{\eta}_{t,I})_{t \in \mathbb{R}}$ form one-parameter groups of automorphisms, so that (locally) we obtain a flow. \square

5 The coset identification for the $N = 2$ super-Virasoro nets with $c < 3$

In this section we prove a crucial result for our analysis. As explained in the introduction, claims for this results have appeared in the literature in the vertex algebraic context, but we could not find any satisfactory and complete proof. The proof is purely operator algebraic in nature but covers the original vertex algebraic statement due to the one-to-one relationship between the two approaches.

Theorem 5.1. *The coset $\mathcal{A}_{\text{U}(1)_{2n+4}} \subset \mathcal{A}_{\text{SU}(2)_n} \otimes \mathcal{A}_{\text{U}(1)_4}$ is completely rational. Its list of irreducible representations is numbered by the following (l, m, s) satisfying $l = 0, 1, 2, \dots, n$, $m = 0, 1, 2, \dots, 2n+3 \in \mathbb{Z}/(2n+4)\mathbb{Z}$, $s = 0, 1, 2, 3 \in \mathbb{Z}/4\mathbb{Z}$ with $l - m + s \in 2\mathbb{Z}$ with the identification $(l, m, s) = (n-l, m+n+2, s+2)$.*

The fusion rules are given as follows, treating the three components l, m, s in the label (l, m, s) separately: For the first component l , we use the usual $SU(2)_n$ fusion rules. For the second component m , we use the group multiplication in $\mathbb{Z}/(2n+4)\mathbb{Z}$. For the third component s , we use the group multiplication in $\mathbb{Z}/4\mathbb{Z}$. All these products are with the identification $(l, m, s) = (n-l, m+n+2, s+2)$.

The conformal spin and dimension of the irreducible DHR sector (l, m, s) are given by

$$\exp\left(\left(\frac{l(l+2)-m^2}{4(n+2)} + \frac{s^2}{8}\right)2\pi i\right), \quad \sin((l+1)\pi/(n+2))/\sin(\pi/(n+2)).$$

Accordingly the statistical dimension is 1 (i.e. we have automorphisms) iff either $l = 0$ or $l = n$.

Proof. This coset net is a special case of coset net studied in [46]. In the notation of [46], this coset net is $\mathcal{A}(G(1, 1, n))$. By (1) of Th. 2.4 in [46], $\mathcal{A}(G(1, 1, n))$ is completely rational. By Th. 4.4 in [46], the Vacuum Pairs in this case is an order two abelian group generated $(n, n+2, 2)$.

Note that this group acts without fixed points on the (l, m, s) as given above. By Th. 4.7 in [46], (l, m, s) are irreducible representations of $\mathcal{A}(G(1, 1, n))$. The rest of the statement in the theorem follows by the remark after Th. 4.7 in [46]. \square

As already mentioned in Theorem 3.2 the unitary representations of the super-Virasoro algebra with central charge $c_n = 3n/(n+2)$ have been explicitly realized by Di Vecchia, Petersen, Yu and Zheng, using the coset construction for the inclusion $U(1)_{2n+4} \subset SU(2)_n \otimes \text{CAR}^{\otimes 2}$ [13]. Now let $\mathcal{A}_{U(1)_{2n+4}} \subset \mathcal{A}_{SU(2)_n} \otimes \mathcal{A}_{\text{CAR}^{\otimes 2}}$ the corresponding inclusion of conformal nets and let \mathcal{C}_n be the corresponding coset net defined by

$$\mathcal{C}_n(I) = \mathcal{A}_{U(1)_{2n+4}}(S^1)' \cap \mathcal{A}_{SU(2)_n}(I) \otimes \mathcal{A}_{\text{CAR}^{\otimes 2}}(I), \quad I \in \mathcal{I}. \quad (5.1)$$

Using the fact that the even part of the NS representation space of $\text{CAR}^{\otimes 2}$ carries the vacuum representation of $U(1)_4$ one can conclude that the bosonic part \mathcal{C}_n^b of the Fermi conformal net \mathcal{C}_n is given by the coset

$$\mathcal{C}_n^b(I) = \mathcal{A}_{U(1)_{2n+4}}(S^1)' \cap \mathcal{A}_{SU(2)_n}(I) \otimes \mathcal{A}_{U(1)_4}(I), \quad I \in \mathcal{I}. \quad (5.2)$$

In analogy with the cases $N = 0$ [34], and $N = 1$ [9] one can show that the results in [13] imply that the $N = 2$ super Virasoro net \mathcal{A}_{c_n} is a covariant irreducible subnet of the coset net \mathcal{C}_n . The aim of this section is to prove that these nets actually coincide, i.e., that $\mathcal{A}_{c_n} = \mathcal{C}_n$ (the $N = 2$ coset identification).

Let us denote by π_m , $m \in \mathbb{Z}/(2n+4)\mathbb{Z}$ the irreducible representations of the net $\mathcal{A}_{U(1)_{2n+4}}$ by π_l , $l = 0, 1, \dots, n$ the irreducible representations of $\mathcal{A}_{SU(2)_n}$ and by π_{NS} the vacuum (Neveu-Schwarz) representation of $\mathcal{A}_{\text{CAR}^{\otimes 2}}$. Then the inclusion $\mathcal{A}_{U(1)_{2n+4}} \otimes \mathcal{C}_n \subset \mathcal{A}_{SU(2)_n} \otimes \mathcal{A}_{\text{CAR}^{\otimes 2}}$ gives decompositions

$$\pi_l \otimes \pi_{NS}|_{\mathcal{A}_{U(1)_{2n+4}} \otimes \mathcal{C}_n} = \bigoplus_{m \in \mathbb{Z}/(2n+4)\mathbb{Z}} \pi_m \otimes \pi_{(l,m)}, \quad (5.3)$$

where $\pi_{(l,m)}$ is the (possibly zero) NS representation of \mathcal{C}_n on the multiplicity space of π_m . The corresponding Hilbert spaces $\mathcal{H}_{l,m}$ carries unitary representations of the NS $N = 2$ super-Virasoro algebra with central charge c_n . Now let

$$\chi_{(l,m)}(t) = \text{tr}_{\mathcal{H}_{(l,m)}} t^{L_0^{\pi_{(l,m)}}} \quad (5.4)$$

be the character of $\pi_{(l,m)}$ (branching function). Although not explicitly stated there the following proposition follows directly from the construction in [13].

Proposition 5.2. *If $|m| \leq l$ and $l + m \in 2\mathbb{Z}$ then the unitary representation of the NS $N = 2$ super-Virasoro algebra on $\mathcal{H}_{l,m}$ with central charge c_n contains a subrepresentation with $h = h_{l,m} := \frac{l(l+2)-m^2}{4(n+2)}$ and $q = q_{n,m} := -\frac{m}{n+2}$. Moreover, $\chi_{(l,m)}(t) = t^{h_{l,m}} + o(t^{h_{l,m}})$ as $t \rightarrow 0^+$, namely $h_{l,m}$ is the lowest conformal energy eigenvalue on $\mathcal{H}_{l,m}$.*

The following lemma will play a crucial role in the proof of the $N = 2$ coset identification.

Lemma 5.3. *If n is even (resp. odd) then the restriction of $\pi_{(n,0)}$ (resp. $\pi_{(n,\pm 1)}$) to \mathcal{A}_{c_n} is irreducible.*

Proof. Let n be even. By Proposition 5.2 the Hilbert space $\mathcal{H}_{(n,0)}$ is a direct sum $\mathcal{K}_1 \oplus \mathcal{K}_2$ where \mathcal{K}_1 carries an irreducible representation of the NS $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2}$ with central charge c_n and lowest energy $h_{(n,0)} = n/4$ and \mathcal{K}_2 is either zero or carries a unitary representation of $\text{SVir}^{N=2,0}$ with central charge c_n and lowest energy $h > n/4$. But $n/4$ is the maximal possible value for the lowest energy and hence $\mathcal{K}_2 = 0$. The case n odd is similar. \square

We are interested in the (NS) DHR sectors of \mathcal{C}_n . They are labeled with (l, m) satisfying $l = 0, 1, 2, \dots, n$, $m = 0, 1, 2, \dots, 2n + 3 \in \mathbb{Z}/(2n + 4)\mathbb{Z}$, with $l - m \in 2\mathbb{Z}$ with the identification $(l, m) = (n - l, m + n + 2)$. The restriction of (l, m) to \mathcal{C}_n^b is given by $(l, m, 0) \oplus (l, m, 2)$. Moreover $(l, m) = \alpha_{(l,m,0)}$ where $\alpha_{(l,m,0)}$ denote the α -induction of $(l, m, 0)$ from \mathcal{C}_n^b to \mathcal{C}_n . In view of Theorem 5.1 then we have the following fermionic fusion rules:

$$(l_1, m_1)(l_2, m_2) = \bigoplus_{\substack{|l_1 - l_2| \leq l \leq \min\{l_1 + l_2, 2n - l_1 - l_2\} \\ l + l_1 + l_2 \in 2\mathbb{Z}}} (l, m_1 + m_2). \quad (5.5)$$

Automorphisms correspond to $l = 0, n$. It follows from Theorem 5.1 and its proof that $[\pi_{(l,m)}] = (l, m)$ for $l - m \in 2\mathbb{Z}$. In particular $[\pi_{(n,0)}] = (n, 0)$ for n even and $[\pi_{(n,\pm 1)}] = (n, \pm 1)$ for n odd where $\pi_{(n,0)}$ and $\pi_{(n,\pm 1)}$ are the representations in Lemma 5.3. Accordingly, for n even, $(n, 0)$ remains irreducible when restricted to \mathcal{A}_{c_n} and similarly, for n odd, $(n, \pm 1)$ remain irreducible when restricted to \mathcal{A}_{c_n} .

Now let \mathcal{A}_c be the $N = 2$ super-Virasoro net with central charge c (not necessarily $c < 3$) and let $J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$ be the corresponding current with Fourier coefficients satisfying the commutation relations

$$[J_n, J_m] = \frac{c}{3} n \delta_{n+m,0}. \quad (5.6)$$

The current $J(z)$ generates a subnet $\mathcal{A}_{U(1)} \subset \mathcal{A}_c$ isomorphic to the $U(1)$ net in [3]. We can label the sectors of $\mathcal{A}_{U(1)}$ by (q) , $q \in \mathbb{R}$, corresponding to $J(z) \mapsto J(z) + qz^{-1}$. They satisfy the DHR fusions $(q_1)(q_2) = (q_1 + q_2)$ see [3] and [48]. Fix an interval $I_0 \in \mathcal{I}_{\mathbb{R}}$. For every $q \in \mathbb{R}$ we choose an endomorphism ρ_q of $\mathcal{A}_{U(1)}$, localized in I_0 and such that $[\rho_q] = (q)$. Now let \mathcal{B} be a graded local extension of \mathcal{A}_c . Following [42] we shall denote by π^0 the vacuum representation of \mathcal{B} , by π_0 the vacuum representation of \mathcal{A}_{c_n} and by $\pi = (\pi^0)^{rest}$ the restriction of π^0 to \mathcal{A}_{c_n} . Now, having the inclusions $\mathcal{A}_{U(1)} \subset \mathcal{A}_c \subset \mathcal{B}$ we can consider the α -inductions (say α^+) $\alpha_{\rho_q}^{\mathcal{A}_c}$ and $\alpha_{\rho_q}^{\mathcal{B}}$ of ρ_q to \mathcal{A}_c and \mathcal{B} respectively. Note that the restriction to \mathcal{A}_c of $\pi^0 \circ \alpha_{\rho_q}^{\mathcal{B}}$ is $\pi \circ \alpha_{\rho_q}^{\mathcal{A}_c}$. Note also that by Theorem 4.3 $\alpha_{\rho_{\frac{q}{3}t}}^{\mathcal{A}_c}$ is unitarily equivalent to the DHR version $\bar{\eta}_t$ of the spectral flow. Accordingly $\alpha_{\rho_{\frac{q}{3}t}}^{\mathcal{B}}$ is a

natural candidate to represent the unitary equivalence class of a possible extension of the *spectral flow* on \mathcal{B} .

Now let $\mathcal{H}_{\mathcal{B}}$ be the vacuum Hilbert space of \mathcal{B} and $\mathcal{H}_{\mathcal{A}_c}$ be the vacuum Hilbert space of \mathcal{A}_c . Since $e^{i\pi J_0}$ is the grading unitary on $\mathcal{H}_{\mathcal{A}_c}$ then $e^{i2\pi J_0} = 1$ on $\mathcal{H}_{\mathcal{A}_c}$ and accordingly the spectrum of J_0 on $\mathcal{H}_{\mathcal{A}_c}$ is contained in \mathbb{Z} . In fact it is not hard to see that this spectrum is exactly \mathbb{Z} . However the spectrum of J_0 on $\mathcal{H}_{\mathcal{B}}$ can be in general larger than \mathbb{Z} even when \mathcal{B} is an irreducible extension of \mathcal{A}_c . This is however not the case if e.g. $e^{i\pi J_0}$ is still the grading unitary on \mathcal{B} , a condition which may be seen as a regularity condition on the extension \mathcal{B} .

Theorem 5.4. *Assume that the spectrum of J_0 on $\mathcal{H}_{\mathcal{B}}$ is \mathbb{Z} . Then, for any $t \in \mathbb{Z}$, $\pi^0 \circ \alpha_{\rho_{\frac{c}{3}t}}^{\mathcal{B}}$ is a NS representation of \mathcal{B} . In particular it restricts to a DHR representation of the bosonic part \mathcal{B}^b of \mathcal{B} .*

Proof. Let Γ be the grading unitary on $\mathcal{H}_{\mathcal{B}}$. We have the spin statistic relation $\Gamma = e^{i2\pi L_0}$. Moreover Γ commutes with the α -induction, namely, for any $I \in \mathcal{I}_{\mathbb{R}}$,

$$\Gamma \pi^0 \circ \alpha_{\rho_{\frac{c}{3}t}}^{\mathcal{B}}(b) \Gamma = \pi^0 \circ \alpha_{\rho_{\frac{c}{3}t}}^{\mathcal{B}}(\Gamma b \Gamma)$$

for all $b \in \mathcal{B}(I)$. To see this let us recall that if $\tilde{I} \in \mathcal{I}_{\mathbb{R}}$ is sufficiently large then there is a unitary $u \in \mathcal{A}_{U(1)}(\tilde{I})$ such that $ubu^* = \alpha_{\rho_{\frac{c}{3}t}}^{\mathcal{B}}(b)$, for all $b \in \mathcal{B}(I)$. and the claim follows from the fact that u commutes with Γ . Now let \mathcal{D} be the subnet covariant subnet of \mathcal{B} defined by

$$\mathcal{D}(I) = \mathcal{A}_{U(1)}(S^1)' \cap \mathcal{B}(I), \quad I \in \mathcal{I}.$$

Note that \mathcal{D} is trivial iff the central charge of the net \mathcal{B} is 1. Then we have the inclusion $\mathcal{A}_{U(1)} \otimes \mathcal{D} \subset \mathcal{B}$. Moreover the subnet $\mathcal{A}_{U(1)} \otimes \mathcal{D}$ contains the Virasoro subnet of \mathcal{B} . The fact that the spectrum of J_0 on $\mathcal{H}_{\mathcal{B}}$ is \mathbb{Z} implies that the restriction of π^0 to $\mathcal{A}_{U(1)}$ is unitarily equivalent to a direct sum of representations ρ_q with $q \in \mathbb{Z}$. Hence the restriction of π^0 to $\mathcal{A}_{U(1)} \otimes \mathcal{D}$ can be written as a direct sum

$$\bigoplus_{q \in \mathbb{Z}} \rho_q \otimes \sigma_q$$

where σ_q is the representation of \mathcal{D} on the (possibly zero) multiplicity space of ρ_q . Accordingly the conformal vacuum Hamiltonian L_0 has the following decomposition

$$L_0 = \bigoplus_{q \in \mathbb{Z}} (L_0^{\rho_q} \otimes 1 + 1 \otimes L_0^{\sigma_q}).$$

Now let us denote $\pi^0 \circ \alpha_{\rho_{\frac{c}{3}t}}^{\mathcal{B}}$ by λ_t . Since $\rho_{\frac{c}{3}t}$ is Möbius covariant, λ_t is a Möbius covariant soliton of \mathcal{B} . Moreover the restriction of λ_t to $\mathcal{A}_{U(1)} \otimes \mathcal{D}$ is

$$\bigoplus_{q \in \mathbb{Z}} \rho_{q+\frac{c}{3}t} \otimes \sigma_q$$

and we have

$$L_0^{\lambda_t} = \bigoplus_{q \in \mathbb{Z}} (L_0^{\rho_{q+\frac{c}{3}t}} \otimes 1 + 1 \otimes L_0^{\sigma_q}).$$

We want to compute the spin operator $e^{i2\pi L_0^{\lambda_t}}$. Recall that the lowest energy in the representation space of ρ_q is given by $\frac{3}{c} \frac{q^2}{2}$, see e.g. [3] (the factor $\frac{3}{c}$ is due to the factor $\frac{c}{3}$

in the commutation relations in (5.6)). It follows that

$$\begin{aligned}
e^{i2\pi L_0^\lambda t} &= \bigoplus_{q \in \mathbb{Z}} e^{i2\pi \frac{3}{2c}(q + \frac{\varepsilon}{3}t)^2} \otimes e^{i2\pi L_0^{\sigma q}} = \bigoplus_{q \in \mathbb{Z}} e^{i2\pi \frac{3}{2c}(q + \frac{\varepsilon}{3}t)^2} \otimes e^{i2\pi L_0^{\sigma q}} \\
&= e^{i2\pi \frac{\varepsilon}{6}t^2} \left(\bigoplus_{q \in \mathbb{Z}} e^{i2\pi \frac{3}{2c}q^2} e^{i2\pi qt} \otimes e^{i2\pi L_0^{\sigma q}} \right) \\
&= e^{i2\pi \frac{\varepsilon}{6}t^2} \left(\bigoplus_{q \in \mathbb{Z}} e^{i2\pi L_0^{\rho q}} e^{i2\pi qt} \otimes e^{i2\pi L_0^{\sigma q}} \right).
\end{aligned}$$

If $t \in \mathbb{Z}$ then $e^{i2\pi qt} = 1$ for all $q \in \mathbb{Z}$. Hence

$$\begin{aligned}
e^{i2\pi L_0^\lambda t} &= e^{i2\pi \frac{\varepsilon}{6}t^2} \left(\bigoplus_{q \in \mathbb{Z}} e^{i2\pi L_0^{\rho q}} \otimes e^{i2\pi L_0^{\sigma q}} \right) \\
&= e^{i2\pi \frac{\varepsilon}{6}t^2} e^{i2\pi L_0} = e^{i2\pi \frac{\varepsilon}{6}t^2} \Gamma
\end{aligned}$$

for all $t \in \mathbb{Z}$. It follows that, for any $t \in \mathbb{Z}$, any $I \in \mathcal{I}_{\mathbb{R}}$ and $b \in \mathcal{B}(I)$, we have

$$e^{i2\pi L_0^\lambda t} \lambda_t(b) e^{-i2\pi L_0^\lambda t} = \lambda_t(\Gamma b \Gamma)$$

and the conclusion follows from Lemma 2.6 □

As pointed out before the spectrum of J_0 on $\mathcal{H}_{\mathcal{B}}$ is in general larger than \mathbb{Z} for an arbitrary extension \mathcal{B} of \mathcal{A}_c . Hence, in particular, the unitary $e^{i\pi J_0}$ does not in general implement the grading of \mathcal{B} . However as a consequence of the following proposition it always implements a gauge automorphism of \mathcal{B} .

Proposition 5.5. *For any $t \in \mathbb{R}$ the operator e^{itJ_0} is a gauge unitary of \mathcal{B} namely $e^{itJ_0} \Omega = \Omega$ and $e^{itJ_0} \mathcal{B}(I) e^{-itJ_0} = \mathcal{B}(I)$ for all $I \in \mathcal{I}$.*

Proof. Obviously we have $e^{itJ_0} \Omega = \Omega$. Now let $I \in \mathcal{I}$ be fixed and let $\tilde{I} \in \mathcal{I}$ be such that the closure of I is contained in \tilde{I} . Chose two real smooth functions on S^1 f_1 and f_2 such that $\text{supp } f_1 \subset \tilde{I}$, $\text{supp } f_2 \subset I'$ and $f_1 + f_2 = 1$. Then $J_0 = J(f_1) + J(f_2)$ on a common core. Hence by locality and the Weyl relations we find

$$e^{itJ_0} \mathcal{B}(I) e^{-itJ_0} = e^{itJ(f_1)} \mathcal{B}(I) e^{-itJ(f_1)} \subset \mathcal{B}(\tilde{I})$$

and since \tilde{I} was an arbitrary interval containing the closure of I we can infer that

$$e^{itJ_0} \mathcal{B}(I) e^{-itJ_0} \subset \mathcal{B}(I) \quad \text{for all } t \in \mathbb{R}$$

and the conclusion follows. □

Now recall from Section 4 that for every $t \in \mathbb{R}$ there is a Möbius covariant soliton $\bar{\eta}_t$ of \mathcal{A}_c corresponding, in the sense of Theorem 4.3 (cf. also Theorem 3.4), to the representation of $\text{SVir}^{N=2,t}$ obtained on the vacuum Hilbert space $\mathcal{H}_{\mathcal{A}_c}$ by the composition of the vacuum representation of the Neveu-Schwarz $N = 2$ super-Virasoro algebra $\text{SVir}^{N=2}$ generating the net \mathcal{A}_c with spectral flow η_t . For $t \in \mathbb{Z}$, $\text{SVir}^{N=2,t}$ coincides with $\text{SVir}^{N=2}$ and, by Theorem 4.3, $\bar{\eta}_t$ is a NS representation of \mathcal{A}_c on $\mathcal{H}_{\mathcal{A}_c}$.

Proposition 5.6. *The representation $\bar{\eta}_1$ of \mathcal{A}_c corresponds to the unitary irreducible representation of $\text{SVir}^{N=2}$ with central charge c and $(h, q) = (\frac{c}{6}, \frac{c}{3})$.*

Proof. It is enough to show that the composition of the vacuum representation with central charge c of $\text{SVir}^{N=2}$ with η_1 is the irreducible representation of $\text{SVir}^{N=2}$ with central charge c and $(h, q) = (\frac{c}{6}, \frac{c}{3})$ on $\mathcal{H}_{\mathcal{A}_c}$. First of all note that the irreducibility of this representation follows from that of the vacuum representation and the invertibility of the spectral flow. Now let $\Omega \in \mathcal{H}_{\mathcal{A}_c}$ be the vacuum vector. Then

$$\begin{aligned}\eta_1(L_m)\Omega &= L_m\Omega + \frac{1}{2}J_m\Omega = 0 \\ \eta_1(J_m)\Omega &= J_m\Omega = 0\end{aligned}$$

for every positive integer m . Moreover,

$$\begin{aligned}\eta_1(G_r^+)\Omega &= G_{r+1}^+\Omega = 0 \\ \eta_1(G_r^-)\Omega &= G_{r-1}^-\Omega = 0\end{aligned}$$

for every positive $r \in \frac{1}{2} + \mathbb{Z}$, where in the second equation we used the fact that $G_{-\frac{1}{2}}^-\Omega = 0$. It follows that Ω is a lowest energy vector also for the representation defined by η_1 and consequently

$$h\Omega = \eta_1(L_0)\Omega = \frac{c}{6}\Omega, \quad q\Omega = \eta_1(J_0)\Omega = \frac{c}{3}\Omega.$$

□

We now come back to the inclusion $\mathcal{A}_{c_n} \subset \mathcal{C}_n$.

Lemma 5.7. $e^{i\pi J_0} = e^{i2\pi L_0} = \Gamma$ on the vacuum Hilbert space $\mathcal{H}_{\mathcal{C}_n}$ of \mathcal{C}_n .

Proof. By Lemma 5.3 there is a NS representation $\tilde{\pi}$ whose restriction to \mathcal{A}_{c_n} is irreducible. Let $\tilde{J}(z) = \sum_{k \in \mathbb{Z}} \tilde{J}_k z^{-k-1}$ be the corresponding current on $\mathcal{H}_{\tilde{\pi}}$. Then

$$e^{i\pi \tilde{J}_0} e^{-i2\pi L_0^{\tilde{\pi}}} \tilde{\pi}_I(x) e^{i2\pi L_0^{\tilde{\pi}}} e^{-i\pi \tilde{J}_0} = \tilde{\pi}_I(e^{i\pi J_0} e^{-i2\pi L_0} x e^{-i2\pi L_0} e^{-i\pi J_0})$$

for all $I \in \mathcal{I}$ and all $x \in \mathcal{C}_n(I)$. In particular

$$e^{i\pi \tilde{J}_0} e^{-i2\pi L_0^{\tilde{\pi}}} \tilde{\pi}_I(x) e^{i2\pi L_0^{\tilde{\pi}}} e^{-i\pi \tilde{J}_0} = \tilde{\pi}_I(x)$$

for all $I \in \mathcal{I}$ and all $x \in \mathcal{A}_{c_n}(I)$ and hence, by irreducibility, $e^{i\pi \tilde{J}_0} e^{-i2\pi L_0^{\tilde{\pi}}}$ must be a multiple of the identity. It follows that

$$\tilde{\pi}_I(e^{i\pi J_0} e^{-i2\pi L_0} x e^{i2\pi L_0} e^{-i\pi J_0}) = \tilde{\pi}_I(x)$$

for all $I \in \mathcal{I}$ and all $x \in \mathcal{C}_n(I)$. Accordingly $e^{i\pi J_0} e^{-i2\pi L_0}$ is also a multiple of the identity and the conclusion follows because $e^{i\pi J_0} e^{-i2\pi L_0} \Omega = \Omega$. □

It follows from Lemma 5.7 that we can apply Theorem 5.4 to the inclusion $\mathcal{A}_{c_n} \subset \mathcal{C}_n$ for any positive integer n . In particular we can conclude that the α -induction $\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}$ of the $U(1)$ automorphism $\rho_{\frac{c_n}{3}}$ is a NS representation of \mathcal{C}_n .

Lemma 5.8. $[\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, -n)$ for every positive integer n .

Proof. $\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}$ is a NS automorphism of the net \mathcal{C}_n and hence $[\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, m)$ for some $m \in \mathbb{Z}$ such that $|m| \leq n$ and $n+m \in 2\mathbb{Z}$. The restriction of $\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}$ to \mathcal{A}_{c_n} is a NS representation of the latter net containing the localized automorphism $\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{A}_{c_n}}$ as a subrepresentation. But the latter is equivalent to η_1 which by Proposition 5.6 corresponds to the representation of $\text{SVir}^{N=2}$ with $(h, q) = (\frac{n}{2(n+2)}, \frac{n}{n+2})$. It follows that $-\frac{m}{n+2} \in \frac{n}{n+2} + \mathbb{Z}$ and hence that $\frac{2-m}{n+2} \in \mathbb{Z}$. Hence, recalling that $|m| \leq n$, we see that either $m = -n$ or $m = 2$. If n is odd $m = 2$ is forbidden. Now let n be even and greater than 2. It follows from Proposition 5.2 that the character in the representation $(n, 2)$ satisfy $\chi_{(n,m)}(t) = t^{\frac{n(n+2)-4}{4(n+2)}} + o(t^{\frac{n(n+2)-4}{4(n+2)}})$ and the equality $[\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, 2)$ would be in contradiction with the fact that in the representation space of $\alpha^{\mathcal{C}_n}$ there must be a nonzero vector with conformal energy $\frac{n}{2(n+2)} < \frac{n(n+2)-4}{4(n+2)}$. Finally if $n = 2$ the equality $[\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, 2)$ would imply that the representation of $\text{SVir}^{N=2}$ corresponding to the restriction to \mathcal{A}_{c_n} of $\alpha_{\rho_{\frac{c_n}{3}}}^{\mathcal{C}_n}$ contains the as subrepresentations the irreducible with $(h, q) = (\frac{1}{4}, -\frac{1}{2})$ and the one with $(h, q) = (\frac{1}{4}, \frac{1}{2})$ in contradiction with $\chi_{(2,2)}(t) = t^{\frac{1}{4}} + o(t^{\frac{1}{4}})$. \square

Lemma 5.9. *For any positive integer k the following hold*

- If $n = 4k$ then $[\alpha_{\rho_{(2k+1)\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, 0)$.
- If $n = 4k - 2$ then $[\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, 0)$.
- If $n = 4k - 1$ then $[\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, -1)$.
- If $n = 4k - 3$ then $[\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] = (n, 1)$.

Proof. We use Lemma 5.8 and the fusion rules in Eq. (5.5). If $n = 4k$ then

$$\begin{aligned} [\alpha_{\rho_{(2k+1)\frac{c_n}{3}}}^{\mathcal{C}_n}] &= (n, -n)^{2k+1} = (n, -(2k+1)n) = (n, -k(2n+4) + 4k - n) \\ &= (n, 0). \end{aligned}$$

If $n = 4k - 2$ then

$$\begin{aligned} [\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] &= (n, -n)^{2k} = (0, -2kn) = (0, -k(2n+4) + n + 2) \\ &= (n, 2n+4) = (n, 0). \end{aligned}$$

If $n = 4k - 1$ then

$$\begin{aligned} [\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] &= (n, -n)^{2k} = (0, -2kn) = (0, -k(2n+4) + n + 1) \\ &= (n, 2n+3) = (n, -1). \end{aligned}$$

If $n = 4k - 3$ then

$$\begin{aligned} [\alpha_{\rho_{2k\frac{c_n}{3}}}^{\mathcal{C}_n}] &= (n, -n)^{2k} = (0, -2kn) = (0, -k(2n+4) + n + 3) \\ &= (n, 2n+5) = (n, 1). \end{aligned}$$

\square

We are now ready to prove the main result of this section

Theorem 5.10. $\mathcal{A}_{c_n} = \mathcal{C}_n$ for every positive integer n .

Proof. If n is a positive integer then, by Lemma 5.3 and Lemma 5.9, there is a $j \in \mathbb{Z}$ such that the restriction to \mathcal{A}_{c_n} of the representation $\alpha_{\rho_j \frac{c_n}{3}}^{\mathcal{C}_n}$ of \mathcal{C}_n is irreducible. Now let π^0 and π_0 be the vacuum representations of \mathcal{C}_n and of \mathcal{A}_{c_n} respectively and let π be the restriction of π^0 to \mathcal{A}_{c_n} . The restriction of $\alpha_{\rho_j \frac{c_n}{3}}^{\mathcal{C}_n}$ to \mathcal{A}_{c_n} is $\pi \circ \alpha_{\rho_j \frac{c_n}{3}}^{\mathcal{A}_{c_n}}$ and since $\alpha_{\rho_j \frac{c_n}{3}}^{\mathcal{A}_{c_n}}$ is an automorphism we can conclude that π is irreducible. Hence $\pi = \pi_0$ and the conclusion follows. \square

6 Classification of $N = 2$ superconformal nets with $c < 3$

We first look at the bosonic part \mathcal{A}^γ of the $N = 2$ super-Virasoro net \mathcal{A}_c , with $c = 3n/(n+2)$ and fixed $n \in \mathbb{N}$, realized as the coset of $\mathrm{SU}(2)_n \otimes \mathrm{U}(1)_2 / \mathrm{U}(1)_{n+2}$ according to Section 5. We use the notation, labelling, and classification and dimension results for the coset net from Theorem 5.1.

The fermionic extension arises from the irreducible DHR sector $(n, n+2, 0)$, which has dimension 1, order 2 and conformal spin -1 .

The modular invariants have been classified by Gannon in [25, Theorem 4]. We are interested in the local extensions of the bosonic part, since its fermionic extension gives an extension of the original fermionic net, so we need to consider only the so-called type I modular invariants. Many of the modular invariants of Gannon's list are not of type I, so we do not need to consider them here. Also, if $(Z_{\lambda\mu})$ is a type I modular invariant, what we need to check is to see whether $\bigoplus_{\lambda} Z_{\lambda\mu} \lambda$ is realized as a dual canonical endomorphism, and if yes, then whether the realization is unique or not. (See [34, Section 4].)

First we deal with the exceptional cases related to the Dynkin diagrams E_6 and E_8 . Consider the case related to E_6 . From Gannon's list of modular invariants, we see that we consider the following three cases for $n = 10$.

(1) The endomorphism $(0, 0, 0) \oplus (6, 0, 0)$. This is a dual canonical endomorphism because this arises from a conformal embedding $\mathrm{SU}(2)_{10} \subset \mathrm{SO}(5)_1$ as in [34, Section 4], which is a special case of a mirror extension studied in [49]. For the same reason as in [34, Section 4] (based on [35]), this realization as a dual canonical endomorphism is unique. (That is, the Q -system is unique up to unitary equivalence.)

(2) The endomorphism $(0, 0, 0) \oplus (0, 12, 0)$. This is a dual canonical endomorphism because this arises from a conformal embedding $\mathrm{U}(1)_{12} \subset \mathrm{U}(1)_3$. The irreducible DHR sector $(0, 12, 0)$ has a dimension 1 and a conformal spin 1, so this is realized as a crossed product by $\mathbb{Z}/2\mathbb{Z}$, and hence unique.

(3) The endomorphism $(0, 0, 0) \oplus (6, 0, 0) \oplus (0, 12, 0) \oplus (6, 12, 0)$. This is a combination of the above two extensions. That is, we first consider an extension in (1) and make another extension as a crossed product by $\mathbb{Z}/2\mathbb{Z}$. For the two above reasons in (1), (2), we conclude this realization of a dual canonical endomorphism is again unique.

The next exceptional case we have to deal with is the case related to the Dynkin diagram E_8 , so we now have $n = 28$.

We have only one modular invariant here and it gives the following.

(4) The endomorphism $(0, 0, 0) \oplus (10, 0, 0) \oplus (18, 0, 0) \oplus (28, 0, 0)$. This arises from a conformal embedding $\mathrm{SU}(2)_{28} \subset (\mathrm{G}_2)_1$, and the realization is unique as in (1).

We now deal with the remaining cases of the modular invariants. From the list of Gannon [25, Theorem 4], we see that when we consider the endomorphisms $\bigoplus_{\lambda} Z_{\lambda\mu} \lambda$ arising from modular invariants $(Z_{\lambda\mu})$, all the endomorphisms λ appearing in this sum have dimensions 1. If all the irreducible DHR sectors of a dual canonical endomorphism have dimension 1, then the extension is a crossed product by a (finite abelian) group and all these irreducible DHR sectors have conformal spin 1. Hence it is enough to consider only irreducible DHR sectors of conformal spin 1 and check whether we can construct an extension using them as a crossed product or not. We divide the cases depending on n .

We start a general consideration. We first look at the irreducible DHR sectors with dimension 1. This condition is equivalent to $l = 0, n$. Then m can be arbitrary in $\{0, 1, \dots, 2n+3\}$, and $s \in \{0, 1\}$ is uniquely determined by the parity of $l+m$, so there are $(4n+8)$ such irreducible DHR sectors. They give an abelian group of order $4n+8$, and we have a simple current extension for each subgroup of this group consisting of irreducible DHR sectors with conformal spin 1, so we need to identify such a subgroup. This subgroup is clearly a subset of all the irreducible DHR sectors with conformal spin 1, but this subset is not a group in general.

If $n \equiv 2 \pmod{4}$, then the irreducible DHR sectors with dimension 1 give the group $(\mathbb{Z}/(2n+4)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, which is generated by $\sigma = (0, 1, 1)$ of order $2n+4$ and $\tau = (n, 0, 0)$ of order 2.

If $n \equiv 0 \pmod{4}$, then the irreducible DHR sectors with dimension 1 give the group $(\mathbb{Z}/(2n+4)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, which is generated by $\sigma = (0, 1, 1)$ of order $2n+4$ and $\tau = (0, n+2, 0)$ of order 2.

If n is odd, the irreducible DHR sectors with dimension 1 give the group $\mathbb{Z}/(4n+8)\mathbb{Z}$, which is generated by $\sigma = (0, 1, 1)$ of order $4n+8$.

The subgroup used for a simple current extension must be a subgroup of these groups. If n is odd, it is clearly a cyclic group. If n is even, it must be a cyclic group or a cyclic group times the cyclic group of order 2. If the latter happens, the subgroup must be of the form

$$G \times (\mathbb{Z}/2\mathbb{Z}) \subset (\mathbb{Z}/(2n+4)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

where G is a subgroup of $\mathbb{Z}/(2n+4)\mathbb{Z}$. This means that the generator for the second component $\mathbb{Z}/2\mathbb{Z}$ must have conformal spin equal to 1, but we see that this is not the case for $n \equiv 0, 2 \pmod{4}$. (In both cases, the conformal spin of the generator for the second component $\mathbb{Z}/2\mathbb{Z}$, which is τ in the above notation, is -1 .) So also for the case of even n , the subgroup used for a simple current extension must be cyclic.

We next note that if the conformal spin of (l, m, s) is 1 with $s = 1$, then n must be a multiple of 16 for the following reason. Suppose the irreducible DHR sector $(l, m, 1)$ has dimension 1 and conformal spin 1. We have $l = 0, n$, and first suppose $l = 0$. Then we have $-2m^2 + (n+2) \in 8(n+2)\mathbb{Z}$ and m is odd. Then we first see n is even, so we set $n = 2a$. Then we have $-m^2 + a + 1 \in 8(a+1)\mathbb{Z}$. Since m is odd, we know $-m^2 + 1 = 0 \pmod{8}$. This implies a is a multiple of 8, hence n is a multiple of 16. Similarly, we now consider the case $l = n$. Then we have $2n(n+2) - 2m^2 + (n+2) \in 8(n+2)\mathbb{Z}$ and $m+n$ is odd. This first gives n is even, so we again set $n = 2a$. Then we have $4a(a+1) - m^2 + a + 1 \in 8(a+1)\mathbb{Z}$ and m is odd. We again have $-m^2 + 1 = 0 \pmod{8}$, so we have that a is a multiple of 8. That is, if n is not a multiple of 16, we need to consider only the irreducible DHR sectors $(l, m, 0)$.

We use the above notations σ, τ for the irreducible DHR sectors of dimensions 1, and find the maximal cyclic subgroup which gives a simple current extension. In general, its any (cyclic) subgroup also works.

We now consider the following four cases one by one.

[A] Case $n \not\equiv 0 \pmod{2}$.

We need to consider only the irreducible DHR sectors $(l, m, 0)$ with $l + m = 0 \pmod{2}$. This shows that we need to consider only the even powers of $\sigma = (0, 1, 1)$. Note that $\sigma^2 = (n, n + 4, 0)$ has the conformal spin $\exp(2\pi ni/(2n + 4))$. Then the conformal spin of σ^{2a} is $\exp(2\pi a^2 ni/(2n + 4))$. Consider the set G consisting of σ^{2a} with conformal spin 1. We show that this set G is a group. Suppose σ^{2a} and σ^{2b} are in G . Then we have $a^2 n/(2n + 4)$ and $b^2 n/(2n + 4)$ are integers, and we need to show $(a + b)^2 n/(2n + 4)$ is also an integer. It is enough to show that $2abn/(2n + 4)$ is an integer. Let j_1, j_2, j_3, j_4 be the numbers of the prime factor 2 in $a, b, n, (n + 2)$, respectively. We then have $2j_1 + j_3 \geq 1 + j_4$, $2j_2 + j_3 \geq 1 + j_4$ and these imply $j_1 + j_2 + j_3 \geq 1 + j_4$. For an odd prime factor p , we apply a similar argument, and we conclude that $2abn/(2n + 4)$ is an integer. Since this set G is a subset of a finite group, this also shows that G is closed under the inverse operation, so it is a subgroup of the cyclic group generated by σ^2 .

We summarize these arguments as follows. We look for the smallest positive integer k with conformal spin of σ^k equal to 1. (Such k is automatically even.) Then the maximal cyclic subgroup giving a simple current extension is $\{1, \sigma^k, \sigma^{2k}, \dots, \sigma^{4n+8-k}\}$.

[B] Case $n = 2 \pmod{4}$.

We need to consider only $(l, m, 0)$ with $l = 0, n$ and $m = 0, 2, 4, \dots, 2m + 2$. If $l = 0$, then the irreducible DHR sector $(l, m, 0)$ has a conformal spin 1 if and only if $m^2/4(n + 2)$ is an integer. If $l = n$, then the irreducible DHR sector $(l, m, 0)$ has a conformal spin 1 if and only if $m^2/4(n + 2) + 1/2$ is an integer. Consider the set G consisting of $(0, 2m, 0)$ with conformal spin 1. As in the argument in case [A], this G is a subgroup of the cyclic group generated by $(0, 2, 0)$. If $(n, m, 0)$ has a conformal spin 1, then $(0, m, 0)$ has a conformal spin -1 and thus $(0, 2m, 0)$ has a conformal spin 1, so this is in G . It means now that we need to consider only the odd powers of the irreducible DHR sector $(n, k/2, 0)$, where k be the smallest even integer with $(0, k, 0)$ having the conformal spin 1. The conformal spin of $(0, k/2, 0)$ must be one of $-1, i, -i$, since the conformal spin of $(0, k, 0)$ is 1. If it is $\pm i$, then all the odd powers of $(0, k/2, 0)$ have conformal spin $\pm i$. If it is -1 , all the odd powers of $(0, k/2, 0)$ have conformal spin -1 . Since the conformal spin of $(n, 0, 0)$ is -1 , in the former case, the maximal cyclic group giving a simple current extension is $\{(0, 0, 0), (0, k, 0), (0, 2k, 0), \dots, (0, 2n + 4 - k, 0)\}$, and in the latter case, the maximal cyclic group giving a simple current extension is

$$\{(0, 0, 0), (n, k/2, 0), (0, k, 0), (n, 3k/2, 0), \dots, (n, 2n + 4 - k/2, 0)\}.$$

[C] Case $n = 4, 8, 12 \pmod{16}$.

First note that both irreducible DHR sectors $(0, 0, 0)$ and $(n, 0, 0)$ have conformal spin 1.

As in [A], the set G of the irreducible DHR sectors $(0, m, 0)$ having a conformal spin 1 is a subgroup of the cyclic group $\mathbb{Z}/(2n + 4)\mathbb{Z}$. Let k be the smallest positive integer such that $k^2/(4n + 8)$ is an integer. Then the group G is given by $\{(0, 0, 0), (0, k, 0), (0, 2k, 0), \dots, (0, 2n + 4 - k, 0)\}$. Then the maximal group giving a simple current extension

$$\{(0, 0, 0), (n, 0, 0), (0, k, 0), (n, k, 0), \dots, (0, 2n + 4 - k, 0), (n, 2n + 4 - k, 0)\}.$$

Note that this group is isomorphic to $\mathbb{Z}/((2n + 4)/k)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but by a general remark above, this also must be a cyclic group. This shows that $(2n + 4)/k$ is always odd.

[D] Case $n = 0 \pmod{16}$.

First note that the conformal spin of $\sigma = (0, 1, 1)$ is $\exp(\pi ni/(4n + 8))$. The set G consisting of powers of σ with conformal spin 1 is again a group as in [A]. Let k be the smallest positive integer such that σ^k has a conformal spin 1. Then G is $\{1, \sigma^k, \sigma^{2k}, \dots, \sigma^{2n+4-k}\}$, where 1 stands for the identity sector $(0, 0, 0)$. We first show that this k is odd. Indeed, the condition that the conformal spin is 1 implies that $nk^2/(8(n + 2))$ is an integer. Now

$n/16$ is an integer, and n and $n/2 + 1$ are relatively prime, so $k^2/(n/2 + 1)$ is also an integer. Since $n/2 + 1$ is odd and k is the smallest such positive integer, we know that k is odd.

If $\sigma^t\tau$ has a conformal spin 1, then $2t$ must be in the set $\{0, k, 2k, \dots, 2n + 4 - k\}$. Since k is odd, we have $t \in \{0, k, 2k, \dots, 2n + 4 - k\}$. The conformal spin of $\sigma^k\tau$ is

$$\exp\left(2\pi i \left(\frac{-(k+n+2)^2}{4(n+2)} + \frac{k^2}{8}\right)\right) = 1,$$

since k is odd, $n = 0 \pmod{16}$ and $nk^2/(8n+16) \in \mathbb{Z}$. All its powers also have a conformal spin 1. We also compute that the conformal spin of $\sigma^{2ak}\tau$ is equal to

$$\exp\left(2\pi i \left(\frac{-(2ak+n+2)^2}{4(n+2)} + \frac{4a^2k^2}{8}\right)\right) = -1,$$

since $n = 0 \pmod{16}$ and $nk^2/(8n+16) \in \mathbb{Z}$. All these together show that the set of the irreducible DHR sectors having conformal spin 1 is

$$\{1, \sigma^k, \sigma^{2k}, \dots, \sigma^{2n+4-k}\} \cup \{\sigma^k\tau, \sigma^{3k}\tau, \dots, \sigma^{2n+4-k}\tau\}.$$

Note that this set is not a group. The maximal subgroup giving a simple current extension is $\{1, \sigma^k, \sigma^{2k}, \dots, \sigma^{2n+4-k}\}$ or $\{1, \sigma^k\tau, \sigma^{2k}\tau, \sigma^{3k}\tau, \dots, \sigma^{2n+4-k}\tau\}$.

We have a few remarks concerning some of the above cases. First we note that if n is odd and the conformal spin for (l, m, s) is 1, we have $l = s = 0$. This shows that all extensions are mirror extensions arising from extensions of $U(1)_{n+2}$.

Second, $n = 4, 8, 12 \pmod{16}$, then all the extensions come from the cosets of the index 2 extensions of $SU(2)_n$, the mirror extensions arising from extensions of $U(1)_{n+2}$ and the combinations of the two.

Finally, if $n = 2 \pmod{4}$, then all the extensions come from the mirror extensions arising from extensions of $U(1)_{n+2}$ and their combinations with $(n, 0, 0)$ which has conformal spin -1 . For example, if $n = 6$, the irreducible DHR sector $(6, 4, 0)$ has dimension 1 and conformal spin 1, and it generates the cyclic group of order 4. These four irreducible DHR sectors are all that have conformal spin 1. This is the second case of [B]. Note that the irreducible DHR sectors $(6, 0, 0)$ and $(0, 4, 0)$ both have conformal spin -1 , so they do not give a coset construction or a mirror extension, but their combination $(6, 4, 0)$ gives a conformal spin 1. The case $n = 10$ gives the first case of [B].

As an example, consider the case $n = 32$. The smallest k with conformal spin of σ^k equal to 1 is 17. The conformal spin of $\sigma^{17}\tau$ is also 1, so we have two maximal subgroups of order 4, $\{1, \sigma^{17}, \sigma^{34}, \sigma^{51}\}$ and $\{1, \sigma^{17}\tau, \sigma^{34}\tau, \sigma^{51}\tau\}$. The union of the two subgroups give a subset of six elements, and this set gives all the irreducible DHR sectors of dimension 1 and conformal spin 1.

Theorem 6.1. *The complete list of $N = 2$ superconformal nets with $c < 3$ in the discrete series is given as follows.*

- A simple current extension arising from a subgroup of the maximal cyclic subgroup appearing in the above [A], [B], [C] and [D].
- The exceptionals related to E_6 and E_8 as in the above (1), (2), (3) and (4).

We finally remark that a cyclic group of an arbitrary order appears in the above classification.

Suppose an arbitrary positive integer j is given. We show that the cyclic group of order j appears in the above [A], [B], [C] and [D]. (This group is not necessarily maximal.)

We may assume $j \neq 1$. Set $n = j^2 - 2$. We are in Case [A] or [B], if j is odd or even, respectively. The irreducible DHR sector $(0, 2j, 0)$ has a conformal spin 1, and the cyclic group $\{(0, 0, 0), (0, 2j, 0), (0, 4j, 0), \dots, (0, 2n + 4 - 2j)\}$ gives a simple current extension. The order of this group is $(2n + 4)/2j = j$.

7 Nets of spectral triples

Next we study the supersymmetry properties of \mathcal{A}_c . Since we would like to construct spectral triples, we may treat this point in a similar way as in [7, Sect.4] for the $N = 1$ super-Virasoro algebra.

First, we recall from Proposition 2.8 that a supercharge (i.e. an odd self-adjoint square-root Q_π of $L_0^\pi - \text{const}$) exists iff π is a Ramond representation though it need not be unique. To be clear we write π_R whenever we deal with a Ramond representation. Then

$$Q_{\pi_R, s} := \cos(s)G_0^{1, \pi_R} + \sin(s)G_0^{2, \pi_R}, \quad s \in \mathbb{R},$$

are, in fact, possible choices satisfying $Q_{\pi_R, s}^2 = L_0^{\pi_R} - c/24$, as can be checked easily by means of Definition 3.1. Moreover, $J_0^{\pi_R}$ acts by rotating this supercharge:

$$e^{i s J_0^{\pi_R}} Q_{\pi_R, 0} e^{-i s J_0^{\pi_R}} = Q_{\pi_R, s}, \quad s \in \mathbb{R}. \quad (7.1)$$

basically a consequence of (4.5). Note, however, that these $Q_{\pi_R, s}$ act on a net defined in the representation (π_R, \mathcal{H}_R) instead of the vacuum representation, i.e., based on (unbounded) generators $L^{\pi_R}(f)$, $G^i, \pi_R(f)$, and $J^{\pi_R}(f)$, and in this sense we shall also perform our domain analysis in the present section. The relation to our actual net \mathcal{A}_c defined in (3.2) is then given by Theorem 3.4.

Second, since $Q_{\pi_R, s}^2 = L_0^{\pi_R} - c_{\pi_R}/24$, the index of the restrictions $Q_{\pi_R, s}|_{\mathcal{H}_{R,+}}$ of the supercharge to the even subspace of $\mathcal{H}_{R,+} \subset \mathcal{H}_R$ is 1 if $h_{\pi_R} = c_{\pi_R}/24$ and 0 otherwise.

As we shall need both the first and the second property, we restrict ourselves henceforth to irreducible graded Ramond representations with $h_{\pi_R} = c_{\pi_R}/24$ (i.e., a vacuum Ramond representation). Note, however, that the present section does not make use of the condition $h_{\pi_R} = c_{\pi_R}/24$. Let π_R then be one fixed such representation, and let $s = 0$, so our supercharge will be $Q_{\pi_R} := G_0^{1, \pi_R}$. Associated to Q_{π_R} , we have in a natural manner a superderivation $(\delta_{\pi_R}, \text{dom}(\delta_{\pi_R}))$ on $B(\mathcal{H}_R)$ as in [7, Sect.2]. In general, it is difficult to decide whether the local domains $\text{dom}(\delta_{\pi_R}) \cap \pi_R(\mathcal{A}_c(I))$ are nontrivial. In [7, Sect.4] we discussed this point for the $N = 1$ super-Virasoro net, and we shall perform a similar procedure here for the $N = 2$ super-Virasoro net \mathcal{A}_c .

Theorem 7.1. *For every $I \in \mathcal{I}$, the subalgebra $\pi_R^{-1}(\text{dom}(\delta_{\pi_R})) \cap \mathcal{A}_c(I) \subset \mathcal{A}_c(I)$ is weakly dense and a local C^* -algebra. It contains in particular the elements*

$$(L(f_1) + J(f_2) + \lambda)^{-1}, \quad J(f_2)(L(f_1) + J(f_2) + \lambda)^{-1}, \quad G^i(f_3)(L(f_1) + J(f_2) + \lambda)^{-1},$$

if $f_1, f_2, f_3 \in C^\infty(S^1)_I$ are positive with $f_1 - C f_2^2, f_1 - C f_3^2 \geq 0$ for some $C > 0$ and $\lambda \in \mathbb{C}$ with $|\Im \lambda|$ sufficiently large.

The proof is quite lengthy and will be subdivided into several lemmata. However, since it borrows many ideas from [7, Sec.4], we can shorten it a bit. For the sake of readability, we drop the superscript and subscript π_R of $Q, c, h, J(f), G^i(f), L(f)$ since we are always working in that fixed Ramond representation, and by \mathcal{A}_c we denote the net generated in that representation instead of the vacuum representation. Working with this new net will be useful throughout the proof, and its relation to the original one in the vacuum representation is established by the lifting property of Theorem 3.4.

Proposition 7.2 ([45, Sec.2]). *Let A be a selfadjoint operator on \mathcal{H} and $(\cdot, \cdot)_k$ the scalar product on $\text{dom}(A^k) \subset \mathcal{H}$ given by*

$$(\psi_1, \psi_2)_k \equiv \langle A^k \psi_1, A^k \psi_2 \rangle.$$

With this scalar product $\text{dom}(A^k)$ is a Hilbert space which we shall denote by \mathcal{H}^k . Assume that $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{N}_0} \text{dom}(A^k) \subset \mathcal{H}$ is dense. Let X, Y be selfadjoint operators on \mathcal{H} such that, for every $k \in \mathbb{N}_0$, there are $C(X, k), C_A(X, k) > 0$ with

$$\|X\psi\|_k \leq C(X, k+1)\|\psi\|_{k+1}, \quad \|[A, X]\psi\|_k \leq C_A(X, k+1)\|\psi\|_{k+1}, \quad \psi \in \mathcal{H}^\infty,$$

and the same for Y in place of X . Then, for every $k \in \mathbb{N} \cup \infty$,

(1) *X is a closable essentially selfadjoint map $\mathcal{H}^k \rightarrow \mathcal{H}^k$ with selfadjoint closure \bar{X} , and the unitary $e^{i\bar{X}}$ defines a bounded map $\mathcal{H}^k \rightarrow \mathcal{H}^k$ with $\|e^{i\bar{X}}\|_k \leq e^{2kC_A(X, k)}$;*

(2) *we have*

$$\|(e^{i\bar{X}} - e^{i\bar{Y}})\psi\|_k \leq C(X - Y, k+1) e^{2(k+1)(C_A(X, k+1) + C_A(Y, k+1))} \|\psi\|_{k+1},$$

in particular, $t \mapsto e^{it\bar{X}}$ is strongly continuous on \mathcal{H}^k .

Moreover, we should remember in this context the linear energy bounds (3.1), (7.2), and the fact that $L(f)$ is bounded from below if f is nonnegative.

Lemma 7.3. *For every $k \in \mathbb{N}$ and real $f_1, f_2 \in C^\infty(S^1)$ and for $\lambda \in \mathbb{C}$ with $|\Im \lambda|$ sufficiently large, we have*

$$(L(f) + \lambda)^{-1} \text{dom}(L_0^k) \subset \text{dom}(L_0^k)$$

and

$$(L(f_1) + J(f_2) + \lambda)^{-1} \text{dom}(L_0^k) \subset \text{dom}(L_0^k).$$

In particular, these resolvents preserve the vector subspace $C^\infty(S^1) \subset \mathcal{H}$ of finite energy vectors.

Proof. We would like to apply the preceding proposition with the positive operator $A = \mathbf{1} + L_0$. Then the linear energy bounds (3.1) show the existence of constants $C(X, k), C_{\mathbf{1}+L_0}(X, k)$, for $k = 0$ and for every X which is the selfadjoint closure of a linear combination of operators of the kind $J(f), L(f), G^i(f)$, with $f \in C^\infty(S^1)_I$. Recall from Section 3 that these X have $C^\infty(L_0) = \mathcal{H}^\infty \subset \mathcal{H}$ as a common invariant core. Together with the commutation relations in Definition 3.1, a similar reasoning as in the case $k = 0$ implies the existence of constants $C(X, k), C_{\mathbf{1}+L_0}(X, k)$, for every $k \in \mathbb{N}_0$, as in Proposition 7.2.

We consider the two cases $X = L(f_1)$ and $X = L(f_1) + J(f_2)$ – the latter one being actually the closure of the sum defined on the common invariant core $C^\infty(L_0)$ as in (4.3), a short-hand notation we shall use always throughout this section. Recall the integral representation of the resolvents as

$$(X + \lambda)^{-1} = -i \int_0^\infty e^{itX} e^{it\lambda} dt, \quad \Im \lambda \neq 0.$$

For $\psi \in \mathcal{H}_k$, the map $t \mapsto e^{itX} e^{it2\lambda} \psi \in \mathcal{H}^k$ is strongly continuous according to Proposition 7.2, and

$$\int_0^\infty \|e^{itX} e^{it\lambda} \psi\|_k dt \leq \|\psi\|_k \int_0^\infty |e^{t(i\lambda + C_{\mathbf{1}+L_0}(X, k))}| dt < \infty.$$

Hence, if $\Im \lambda > C_{\mathbf{1}+L_0}(X, k)$, then $(X + \lambda)^{-1}\psi \in \mathcal{H}^k$, and, by taking the adjoint of the above, we obtain the same statement for $\Im \lambda < -C_{\mathbf{1}+L_0}(X, k)$. \square

From now on we shall content ourselves with the most general case $k = \infty$ and work on the vector subspace $\mathcal{H}^\infty = C^\infty(L_0)$, cf. Section 3.

Lemma 7.4. *Let f_1 and f_2 be real smooth functions on S^1 and assume that $f_1^2 \leq C f_2$ for some $C > 0$. Then, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

$$G^i(f_1)(L(f_2) + \lambda)^{-1} \in B(\mathcal{H}).$$

The proof goes as in [7, Prop.4.6].

Lemma 7.5. *Let $f \in C^\infty(S^1)_I$ with $f \geq 0$, and $\lambda \in \mathbb{C}$ with $|\Im \lambda|$ sufficiently large. Then $(L(f) + \lambda)^{-1} \in \text{dom}(\delta) \cap \mathcal{A}_c(I)$, and*

$$\delta((L(f) + \lambda)^{-1}) = -\frac{i}{2}(L(f) + \lambda)^{-1}G^1(f')(L(f) + \lambda)^{-1},$$

Proof. According to Lemma 7.3, $(L(f) + \Re \lambda + i\beta)^{-1}\psi \in C^\infty(L_0)$, for any $\psi \in C^\infty(L_0)$ and $\beta > 0$ sufficiently large. The commutation relations for G then imply $L(f)(L(f) + \Re \lambda + i\beta)^{-1}\psi \in C^\infty(L_0)$ and

$$QL(f)(L(f) + \Re \lambda + i\beta)^{-1}\psi = L(f)Q(L(f) + \Re \lambda + i\beta)^{-1}\psi + \frac{i}{2}G(f')(L(f) + \Re \lambda + i\beta)^{-1}\psi.$$

Adding $i\beta Q(L(f) + \Re \lambda + i\beta)^{-1}\psi$ to both sides of the previous equality we find

$$Q\psi = (L(f) + \Re \lambda + i\beta)Q(L(f) + \Re \lambda + i\beta)^{-1}\psi + \frac{i}{2}G(f')(L(f) + \Re \lambda + i\beta)^{-1}\psi.$$

We then let $(L(f) + \Re \lambda + i\beta)^{-1}$ act on both sides of the latter equality. The fact that $(L(f) + \Re \lambda + i\beta)^{-1}G^1(f')(L(f) + \Re \lambda + i\beta)^{-1}$ is bounded follows from Lemma 7.4 together with $f'^2 \leq Cf$ for a suitable constant $C > 0$ using e.g. l'Hôpital's rule. \square

A similar procedure works for $(L(f_1) + J(f_2) + \lambda)^{-1}$.

Lemma 7.6. *Let $f \in C^\infty(S^1)_I$ and $\lambda \in \mathbb{C}$ with imaginary part sufficiently large. Then*

$$G^i(f)(L(f) + \lambda)^{-1}, \quad i = 1, 2,$$

are in $\text{dom}(\delta) \cap \mathcal{A}_c(I)$, and

$$\begin{aligned} \delta(G^1(f)(L(f) + \lambda)^{-1}) &= \left(2L(f) - \frac{c}{24\pi} \int_{S^1} f\right)(L(f) + \lambda)^{-1} \\ &\quad + \frac{i}{2}G^1(f)(L(f) + \lambda)^{-1}G^1(f')(L(f) + \lambda)^{-1}, \\ \delta(G^2(f)(L(f) + \lambda)^{-1}) &= -J(f')(L(f) + \lambda)^{-1} \\ &\quad + \frac{i}{2}G^2(f)(L(f) + \lambda)^{-1}G^1(f')(L(f) + \lambda)^{-1}. \end{aligned}$$

Proof. From Lemma 7.5 we know that, for $\psi \in C^\infty(L_0)$,

$$Q(L(f) + \lambda)^{-1}\psi = (L(f) + \lambda)^{-1}Q\psi - \frac{i}{2}(L(f) + \lambda)^{-1}G(f')(L(f) + \lambda)^{-1}\psi.$$

From the fact that $(L(f) + \lambda)^{-1}\psi \in C^\infty(L_0)$ and $Q\psi \in C^\infty(L_0)$ we have that $Q(L(f) + \lambda)^{-1}\psi$ and $(L(f) + \lambda)^{-1}Q\psi$ belong to $C^\infty(L_0)$. Hence

$$(L(f) + \lambda)^{-1}G^i(f')(L(f) + \lambda)^{-1}\psi \in C^\infty(L_0).$$

The domain properties and relations for $i = 1$ are obtained precisely as in [7, Th.4.11] because of our choice $Q = G_0^1$ here.

Concerning the case $i = 2$, we get

$$\begin{aligned} QG^2(f)(L(f) + \lambda)^{-1}\psi &= -G^2(f)(L(f) + \lambda)^{-1}Q\psi \\ &\quad + \frac{i}{2}G^2(f)(L(f) + \lambda)^{-1}G^1(f')(L(f) + \lambda)^{-1}\psi \\ &\quad - J(f')(L(f) + \lambda)^{-1}\psi. \end{aligned}$$

We know from the assumptions of this lemma and Lemma 7.4 that, if for some $C > 0$, $f_1 - Cf_2^2$ is positive, then $G^i(f_2)(L(f_1) + \lambda)^{-1}$ is bounded. This shows that the one but last term extends to a bounded operator on \mathcal{H} . By making use of local energy bounds (7.2) for currents as we shall explain below in the proof of Lemma 7.8, we obtain a bound on the last term, too, so the lemma is proved. \square

With a similar reasoning we obtain

Lemma 7.7. *Let $f_1, f_2 \in C^\infty(S^1)_I$ with $f_1 - Cf_2^2 \geq 0$ for some $C > 0$, and let $\lambda \in \mathbb{C}$ with imaginary part sufficiently large. Then $J(f_2)(L(f_1) + \lambda)^{-1} \in \text{dom}(\delta) \cap \mathcal{A}_c(I)$, and*

$$\begin{aligned} \delta(J(f_2)(L(f_1) + \lambda)^{-1}) &= -iG^2(f_2)(L(f_1) + \lambda)^{-1} \\ &\quad + J(f_2)(L(f_1) + \lambda)^{-1}\frac{i}{2}G^1(f_1')(L(f_1) + \lambda)^{-1}. \end{aligned}$$

Lemma 7.8. *Let $f_1, f_2 \in C^\infty(S^1)_I$ with $f_1 - Cf_2^2 \geq 0$ for some $C > 0$, and let $\lambda \in \mathbb{C}$ with imaginary part sufficiently large. Then*

$$(L(f_1) + J(f_2) + \lambda)^{-1} \text{ and } J(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}$$

are in $\text{dom}(\delta) \cap \mathcal{A}_c(I)$, and

$$\begin{aligned} &\delta((L(f_1) + J(f_2) + \lambda)^{-1}) \\ &= (L(f_1) + J(f_2) + \lambda)^{-1}\left(\frac{i}{2}G^1(f_1') - iG^2(f_2)\right)(L(f_1) + J(f_2) + \lambda)^{-1} \\ &\delta(J(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}) \\ &= -iG^2(f_1)(L(f_1) + J(f_2) + \lambda)^{-1} \\ &\quad + J(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}\left(\frac{i}{2}G^1(f_1') - iG^2(f_2)\right)(L(f_1) + J(f_2) + \lambda)^{-1}. \end{aligned}$$

Notice that $f_2 = C'f_1'$, with suitable $C' \in \mathbb{R}$, fulfils the assumptions and will be of particular interest.

For the proof we shall need *local energy bounds* [11]: instead of bounding $J(f)$, for $f \in C^\infty(S^1)$, by a multiple of $\mathbf{1} + L_0$ as in (3.1), it may be bounded by smeared fields, namely

$$\frac{3}{2c_\pi}J(f)J(f) \leq r_f\mathbf{1} + L(f^2), \quad r_f \text{ some positive constant.} \quad (7.2)$$

This holds in every irreducible representation, and we are of course interested in π_R in the present section.

Proof. The fact that both families of operators preserve $C^\infty(L_0)$ has been obtained in Lemma 7.3. Here we first show that for sufficiently large $C_3 > 0$ (depending only on f_1, f_2), we have

$$\|(L(f_1) + J(f_2) + C_3)\psi\| \geq \|J(f_2)\psi\|, \quad \psi \in C^\infty(L_0).$$

Using the local energy bounds in (7.2) we see $J(f_2)^2 \leq C_1 \mathbf{1} + (2c/3)L(f_2^2)$, with a constant $C_1 > 0$ depending only on f_2 and not on the representation in which we are working. This is again bounded by $(2c/3C)L(f_1) + 2C_2$ with constant $C_2 > 0$ depending on f_1, f_2, C and C_1 such that $(2c/3C)L(f_1) + C_2 \mathbf{1} > \mathbf{1}$, so by functional calculus:

$$J(f_2)^2 \leq C_1 \mathbf{1} + \frac{2c}{3}L(f_2^2) \leq \frac{2c}{3C}L(f_1) + C_2 \mathbf{1} \leq \left(\frac{2c}{3C}L(f_1) + 2C_2 \mathbf{1}\right)^2$$

on $C^\infty(L_0)$ because $L(f)$ is bounded from below for every nonnegative f according to [22, Th.4.1]. Then we compute

$$\begin{aligned} \|(L(f_1) + J(f_2) + C_3)\psi\|^2 &\geq \|(L(f_1) + C_3)\psi\|^2 - 2\|(L(f_1) + C_3)\psi\|\|J(f_2)\psi\| \\ &\quad + \|J(f_2)\psi\|^2 \\ &\geq \|(L(f_1) + C_3)\psi\|^2 \\ &\quad - 2\|(L(f_1) + C_3)\psi\|\frac{2c}{3C}\|(L(f_1) + CC_2)\psi\| + \|J(f_2)\psi\|^2 \\ &\geq \|(L(f_1) + C_3)\psi\|^2 \\ &\quad - 2\|(L(f_1) + C_3)\psi\|\frac{2c}{3C}\|(L(f_1) + C_3)\psi\| + \|J(f_2)\psi\|^2 \\ &\geq (1 - \frac{4c}{3C})\|(L(f_1) + C_3)\psi\|^2 + \|J(f_2)\psi\|^2 \\ &\geq \|J(f_2)\psi\|^2 \end{aligned}$$

if $C_3 > CC_2$ and $C \geq 4c/3$, so we are done. Rescaling the functions f_1 and f_2 we can actually weaken the latter condition to $C > 0$.

Next, we recall from Lemma 7.3 that $(L(f_1) + J(f_2) + \lambda)^{-1}$ preserves $C^\infty(L_0)$ if $|\Im \lambda|$ is sufficiently large. Substituting then ψ with $(L(f_1) + J(f_2) + \lambda)^{-1}\psi$ in the above, we can compute

$$\begin{aligned} \|J(f_2)(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| &\leq \|(L(f_1) + J(f_2)C_3)(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\leq \|(L(f_1) + J(f_2))(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\quad + \|C_3(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\leq \frac{1 + |\Re \lambda|}{|\Im \lambda|}\|\psi\| + \frac{C_3}{|\Im \lambda|}\|\psi\|. \end{aligned}$$

And finally

$$\begin{aligned} \|L(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| &\leq \|L(f_1) + J(f_2) + \lambda\|(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\quad + \|J(f_2)(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\quad + |\lambda|\|(L(f_1) + J(f_2) + \lambda)^{-1}\psi\| \\ &\leq \|\psi\| + \frac{1 + C_3 + |\Re \lambda|}{|\Im \lambda|}\|\psi\| + \frac{|\lambda|}{|\Im \lambda|}\|\psi\|. \end{aligned}$$

This shows that both $J(f_2)(L(f_1) + J(f_2) + \lambda)^{-1}$ and $L(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}$, and consequently also $G^i(f_1)(L(f_1) + J(f_2) + \lambda)^{-1}$, are bounded operators on \mathcal{H} for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

preserving $C^\infty(L_0)$ if $|\Im \lambda|$ is sufficiently large. The concrete commutation relations are now proved in the same way as Lemmata 7.5 and 7.6, making use of Definition 3.1 which implies the commutation relation $[G_0^1, J(f)] = -iG^2(f)$ and $[G_0^1, L(f)] = \frac{i}{2}G^1(f)$ on $C^\infty(L_0)$. \square

Proof of Theorem 7.1. The proof goes now almost precisely as in [7, Lem.4.12], but for the sake of completeness and because of its importance we present it here again. $\mathcal{A}_c(I) \cap \text{dom}(\delta)$ is a unital $*$ -subalgebra of $\mathcal{A}_c(I)$ wherefore, by the von Neumann density theorem, it suffices to show that

$$(\mathcal{A}_c(I) \cap \text{dom}(\delta))' \subset \mathcal{A}_c(I)'.$$

To this end let f be an arbitrary real smooth function with support in I . Recalling that I must be open it is easy to see that there is an interval $I_0 \in \mathcal{I}$ such that $\overline{I_0} \subset I$ and $\text{supp } f \subset I_0$ and a smooth function g on S^1 such that $\text{supp } g \subset \overline{I_0}$, $g(z) > 0$ for all $z \in I_0$, $g'(z) \neq 0$ for all $z \in I_0$ sufficiently close to the boundary and $g(z) = 1$ for all $z \in \text{supp } f$. Accordingly, there is a real number $s > 0$ such that $f(z) + sg(z) > 0$ for all $z \in I_0$. Now let $f_1 = f + sg$ and $f_2 = sg$. Then $f = f_1 - f_2$. Hence it follows from the above lemmata and the definition of $\mathcal{A}_c(I)$ that, for $|\Im \lambda|$ sufficiently large, all the operators

$$(L(f_j) + \lambda)^{-1}, J(f_j)(L(f_j) + \lambda)^{-1}, G^i(f_j)(L(f_j) + \lambda)^{-1}, \quad j = 1, 2,$$

(and even further ones) belong to $(\mathcal{A}_c(I) \cap \text{dom}(\delta))'$. So if $a \in (\mathcal{A}_c(I) \cap \text{dom}(\delta))'$, then a commutes with $L(f_j)$, $J(f_j)$, and $G^i(f_j)$, $j = 1, 2$. Therefore, if $\psi_1, \psi_2 \in C^\infty(L_0)$ then,

$$\begin{aligned} (a\psi_1, L(f)\psi_2) &= (a\psi_1, L(f_1)\psi_2) - (a\psi_1, L(f_2)\psi_2) \\ &= (aL(f_1)\psi_1, \psi_2) - (aL(f_2)\psi_1, \psi_2) \\ &= (aL(f)\psi_1, \psi_2) \end{aligned}$$

and, since $C^\infty(L_0)$ is a core for $L(f)$, it follows that a commutes with $L(f)$ and hence with $e^{iL(f)}$. Similarly a commutes with $e^{iG^i(f)}$ and $e^{iJ(f)}$. Hence $a \in \mathcal{A}_c(I)'$ and the conclusion follows. It is a general fact (cf. e.g. [7, Prop. 2.2]) that $\text{dom}(\delta) \cap \mathcal{A}_c(I)$ is a local C^* -algebra. \square

Let us now return to the original notation of the net \mathcal{A}_c and fields in the vacuum representation, thus reintroducing the explicit π_R from before. To conclude this section then, recall from [7, Def.3.9]:

Definition 7.9. A *net of graded spectral triples* $(\mathfrak{A}(I), (\pi_I, \mathcal{H}), Q_\pi)_{I \in \mathcal{I}(\mathcal{I}_\mathbb{R})}$ over S^1 (or $S^1 \setminus \{-1\} \simeq \mathbb{R}$, respectively) consists of a graded Hilbert space \mathcal{H} , a selfadjoint operator Q_π , and a net \mathfrak{A} of unital $*$ -algebras on \mathcal{I} (or $\mathcal{I}_\mathbb{R}$) acting on \mathcal{H} via the graded general soliton π , i.e., a map from \mathcal{I} (or $\mathcal{I}_\mathbb{R}$) into the family of unital $*$ -algebras represented on $B(\mathcal{H})$ which satisfies the isotony property

$$\mathfrak{A}(I_1) \subset \mathfrak{A}(I_2) \quad \text{if } I_1 \subset I_2,$$

and such that $(\mathfrak{A}(I), (\pi_I, \mathcal{H}), Q_\pi)$ is a graded spectral triple for all $I \in \mathcal{I}$ (or $I \in \mathcal{I}_\mathbb{R}$).

Corollary 7.10. *Setting $\mathfrak{A}(I) := \mathcal{A}_c(I) \cap \pi_R^{-1}(\text{dom}(\delta^{\pi_R}))$, the family*

$$(\mathfrak{A}(I), (\pi_R, \mathcal{H}_R), Q_{\pi_R})_{I \in \mathcal{I}}$$

forms a nontrivial net of even θ -summable graded spectral triples over S^1 .

8 JLO cocycles, index pairings, and Ramond sectors

General case

In the preceding section, in particular in Corollary 7.10, we constructed in a canonical manner a nontrivial net of spectral triples for a given Ramond representation. They give rise to JLO cocycles and thus define an index pairing as constructed in [8]. There, however, we mentioned that for the local algebras the cocycles in the sectors are all equivalent and therefore unsuitable for our purpose. Moreover, the characteristic projections in [8, Sec.5] are never contained in the local algebras. Therefore we have to turn to a global construction. The only difference to the general construction in [8] is that here we will dispense with differentiable transportability of the sectors as explained below. We proceed here as follows: we define a global algebra, construct global spectral triples, show their non-triviality and some other properties, introduce the characteristic projections, and finally construct and evaluate the associated index pairing for the sectors. For a better understanding of what is going on in the present section we suggest to have a look at [8]; in particular, the reader unfamiliar with noncommutative geometry should do so first.

Let \mathcal{A} be a superconformal net and let $\text{vN}(\mathcal{A}^\gamma)$ be the universal von Neumann algebra from Definition 2.4. Then every locally normal representation of \mathcal{A}^γ can be identified with a locally normal representation of $\text{vN}(\mathcal{A}^\gamma)$ and vice versa.

Definition 8.1. Let Δ_R be a family of mutually inequivalent irreducible graded Ramond representation of \mathcal{A} satisfying the trace-class condition

$$\text{tr}(e^{-sL_0^\pi}) < \infty, \quad s > 0, \quad \pi \in \Delta_R.$$

The associated *differential global algebra* is defined as

$$\mathfrak{A}_{\Delta_R}^0 := \{a \in \text{vN}(\mathcal{A}^\gamma) : (\forall \pi \in \Delta_R) \pi(a) \in \text{dom}(\delta_\pi)\},$$

and $\mathfrak{A}_{\Delta_R}^0(I) := \mathfrak{A}_{\Delta_R}^0 \cap \mathcal{A}(I)$.

We remark that this definition is slightly different from, yet directly related to [8, Def.4.6]

Proposition 8.2. *The differential global algebra $\mathfrak{A}_{\Delta_R}^0$ is nontrivial, but depends sensitively on the family Δ_R .*

Proof. We recall from Theorem 7.1 that in the representations, the resolvents

$$(L^\pi(f_1) + \lambda)^{-1}, \quad G^\pi(f_3)(L^\pi(f_1) + \lambda)^{-1},$$

are contained in $\text{dom}(\delta_\pi) \cap \mathcal{A}^\pi(I)$, for every $\pi \in \Delta_R$, in the case of the $(N = 2)$ -super-Virasoro net. The proof extends however to an arbitrary superconformal net in a supersymmetric representation. In view of Theorem 3.4 this implies that we can pull out the π , and the resolvents

$$(L(f_1) + \lambda)^{-1}, \quad G(f_3)(L(f_1) + \lambda)^{-1},$$

are contained in $\bigcap_{\pi \in \Delta_R} \pi^{-1}(\text{dom}(\delta_\pi)) \cap \mathcal{A}(I)$, whose even part in turn equals $\mathfrak{A}_{\Delta_R}^0(I)$. Obviously, the generated $*$ -algebra is nontrivial.

The *global algebra* $\pi^{-1}(\text{dom}(\delta_\pi)^\gamma) \subset \text{vN}(\mathcal{A}^\gamma)$, however, depends on the precise choice of the class of $\pi \in \Delta_R$. In fact, given $\pi_1 \in \Delta_R$, suppose there is an element $x_1 \in \text{vN}(\mathcal{A}^\gamma)$ such that $\pi_1(x_1) = \mathbf{1}$ while $\pi(x_1) = 0$ for all $\pi \in \Delta_R$ with $\pi \neq \pi_1$. Now choose any element

$y \in \mathcal{A}(I)$ such that $\pi_1(y) \notin \text{dom}(\delta_{\pi_1})$, which is clearly possible since δ_{π_1} is an unbounded superderivation. Then

$$\pi(x_1 y) = \begin{cases} \pi_1(y) \notin \text{dom}(\delta_{\pi_1}) : \pi = \pi_1 \\ 0 \in \text{dom}(\delta_{\pi}) : \text{otherwise,} \end{cases}$$

so $x_1 y \in \pi^{-1}(\text{dom}(\delta_{\pi}))$, but $x_1 y \notin \pi_1^{-1}(\text{dom}(\delta_{\pi_1}))$ if $\pi \neq \pi_1$. The element x_1 will be the characteristic projection of π_1 constructed in Proposition 8.6 below. \square

Combining the preceding proposition with the preceding section and the fact that the representations of the net correspond to those of the global algebra $\text{vN}(\mathcal{A}^\gamma)$ and hence to those of the dense subalgebra $\mathfrak{A}_{\Delta_R}^0$, we obtain

Corollary 8.3. *For every $\pi \in \Delta_R$ and with $Q_\pi := G_0^{1,\pi}$ as before, $(\mathfrak{A}_{\Delta_R}^0, (\pi, \mathcal{H}_\pi, \Gamma_\pi), Q_\pi)$ forms a nontrivial θ -summable graded spectral triple, and the associated JLO cocycle τ_π is well-defined.*

Remark 8.4. (1) The second part is a consequence of the fact that to every θ -summable spectral triple there corresponds an entire cyclic cocycle, the JLO cocycle [33]. We used that cocycle in the construction of the index pairing in [8, Sec.5].

(2) In [8, Sec.5], we fixed one spectral triple $(\mathfrak{A}_\Delta, (\pi_R, \mathcal{H}_{\pi_R}), Q_{\pi_R})$, based on a given family Δ of suitable localized endomorphisms of $C^*(\mathcal{A}^\gamma)$, and hence one JLO cocycle τ and obtained the other cocycles (associated to a sector with “differentiable” representative ρ) as pullback $\rho^* \tau$, which turned actually out to be the JLO cocycles of the spectral triples $(\mathfrak{A}_\Delta, (\pi_R \rho, \mathcal{H}_{\pi_R}), Q_{\pi_R})$ with common Q_{π_R} . In contrast to that construction, we have here *individual* global spectral triples $(\mathfrak{A}_{\Delta_R}^0, (\pi, \mathcal{H}_\pi, \Gamma_\pi), Q_\pi)$ and hence an *individual* JLO cocycle τ_π for every representation $\pi \in \Delta_R$. If $\pi = \pi_R \rho$ for some $\rho \in \Delta$, a relation between the JLO cocycles $\tau_\pi = \tau_{\pi_R \rho}$ and $\rho^* \tau_{\pi_R}$ is possible, but in general the two are not equivalent. The second one, presented in [8], works only for differentially transportable endomorphisms ρ of \mathfrak{A}_Δ : we need a strong compatibility condition between ρ and δ_{π_R} , which we do not have in the $N = 2$ super-Virasoro model. The present construction instead works always. However, we can no longer consider the differentiable sectors as acting on entire cyclic cohomology. In case both constructions work, the resulting cocycles $\tau_{\pi_R \rho}$ and $\rho^* \tau_{\pi_R}$ coincide if $Q_{\pi_R \rho} - Q_{\pi_R}$ is a bounded operator: in this case, one cocycle can be obtained from the other one by a homotopy argument. The two cocycles *do not coincide* if the indices of $Q_{\pi_R \rho}$ and Q_{π_R} are different, because in that case

$$\rho^* \tau_{\pi_R}(\mathbf{1}) = \tau_{\pi_R}(\rho(\mathbf{1})) = \tau_{\pi_R}(\mathbf{1}) = \text{ind}_{\mathcal{H}_{R,+}}(Q_{\pi_R}) \neq \text{ind}_{\mathcal{H}_{R,+}}(Q_{\pi_R \rho}) = \tau_{\pi_R \rho}(\mathbf{1}).$$

\square

In order to compare the several cocycles τ_π , $\pi \in \Delta_R$, we may evaluate them on cycles. The best choice consists of those coming from the so-called characteristic projections in [8, Sec.5]. To be precise, we make the following slightly modified

Definition 8.5. Given a graded Ramond representation $(\pi_R, \mathcal{H}_R, \Gamma_R)$ of \mathcal{A} , write $s(\pi) \in \text{vN}(\mathcal{A}^\gamma)$ for the central support of projection onto the subrepresentation π of the universal representation. Moreover, write $p \in B(\mathcal{H}_\pi)$ for the finite-dimensional projection onto the lowest energy subspace of \mathcal{H}_π . Then we call the associated projection $p_{0,+} := s(\pi) \pi^{-1}(p) \in \text{vN}(\mathcal{A}^\gamma)$ the *characteristic projection* of π .

In other words, for $\pi, \pi' \in \Delta_R$, we have

$$\pi'(p_\pi) = \begin{cases} \text{projection onto lowest energy subspace} : \pi' = \pi \\ 0 : \pi' \neq \pi. \end{cases}$$

The projection p_π is well-defined and lies actually in $\mathfrak{A}_{\Delta_R}^0$, as proved in a slightly modified way in [8, Sec.5].

$N = 2$ super-Virasoro net

From now on, let Δ_R denote the family of all mutually inequivalent irreducible graded vacuum Ramond representations of \mathcal{A}_c satisfying the trace-class condition.

Proposition 8.6. *The characteristic projections of $(\mathcal{A}_c, \Delta_R)$ are given by*

$$p_q = \chi_1(e^{-(L_0 - \frac{3}{2}q^2)})\chi_1(e^{2\pi i(J_0 - q)})\frac{1 + e^{\pi i J_0}}{2} \in \mathfrak{A}_{c, \Delta_R}^0.$$

Proof. The first part on the RHS is the characteristic projection onto the subspace with lowest weight and charge $(c/24, q)$, so $\pi(p_q) = 0$ iff $\pi \neq \pi_{(c/24, q)}$, recalling that all irreducible unitary representations of \mathcal{A}_c are characterised by the tuple (h, q) . Since $\pi(p_q)$ is a subprojection of a finite-dimensional spectral projection of Q_π , for every $\pi \in \Delta_R$, we clearly have $p_q \in \mathfrak{A}_{c, \Delta_R}^0$. The second factor on the RHS is the projection onto the even subspace since the grading in the representation π is implemented by $e^{i\pi J_0^\pi}$ as mentioned above in Lemma 4.6, so we are done. \square

With all these ingredients at hand, we can now prove the main result of this section, which is an adaptation of the index pairing construction in [8] to the case where differentiable transportability fails but the rest continues to hold.

Theorem 8.7. *The index pairing for $(\mathcal{A}_c, \Delta_R)$ in the sectors gives*

$$\tau_\pi(p_{\pi'}) = \begin{cases} 1 : \pi' = \pi \\ 0 : \text{otherwise}, \end{cases}$$

for $\pi, \pi' \in \Delta_R$. Hence, the family $\{p_\pi : \pi \in \Delta_R\} \subset \mathfrak{A}_{c, \Delta_R}^0$ separates Δ_R .

Proof. By definition of Δ_R , the lowest energy subspace is one-dimensional and spanned by an *even* vector, so $\dim(\pi(p_\pi)\mathcal{H}_{\pi,+}) = 1$: in fact, we have $L_0^\pi \Omega_\pi = \frac{c}{24}\Omega_\pi$, $\Gamma_\pi \Omega_\pi = \Omega_\pi$ and $G_0^{\pm, \pi} \Omega_\pi = 0$. Thus, owing to Proposition 8.6, we have

$$\tau_\pi(p_{\pi'}) = \text{ind}_{\pi(p_{\pi'})\mathcal{H}_{\pi,+}}(\pi(p_{\pi'})Q_\pi\pi(p_{\pi'})) = \text{ind}_{\pi(p_{\pi'})\mathcal{H}_{\pi,+}}(0) = \dim(\pi(p_{\pi'})\mathcal{H}_{\pi,+}),$$

and the latter term equals 0 if $\pi' \neq \pi$ and 1 otherwise. \square

Corollary 8.8. *The characteristic projections are mutually orthogonal and commutative, so they generate a finite-dimensional abelian subalgebra of $Z(\text{vN}(\mathcal{A}_c^\gamma))$ isomorphic to \mathbb{C}^{Δ_R} .*

This lets the representation theory of the even subnet appear in still another form of noncommutative geometry, namely the cyclic cohomology of the finite-dimensional algebra \mathbb{C}^{Δ_R} , which in turn equals \mathbb{C}^{Δ_R} . In the following section, we are going to partially relate this finite-dimensional algebra to the chiral ring. Interesting applications of this fact to K-theory in the light of [6] shall be treated elsewhere.

9 Outlook: the chiral ring and fusion rules

As in Section 8, let Δ_R be a maximal family of mutually inequivalent irreducible graded vacuum Ramond localized endomorphisms of the $N = 2$ super-Virasoro net \mathcal{A}_c at a fixed central charge c . For the statements not proven here we refer to [1, Sec.5.2].

Definition 9.1. An irreducible lowest weight Neveu-Schwarz representation π of $\text{SVir}^{N=2}$ with given central charge c is called *chiral* (*antichiral*) if its lowest vector Ω_π satisfies

$$G_{-1/2}^{+, \pi} \Omega_\pi = 0 \quad (G_{-1/2}^{-, \pi} \Omega_\pi = 0, \text{ respectively}).$$

It is called *primary* if

$$G_r^{+, \pi} \Omega_\pi = G_r^{-, \pi} \Omega_\pi = 0, \quad r > 0.$$

We denote the equivalence classes of chiral primary representations by Δ_{cp} .

It turns out that a representation is chiral (antichiral) primary iff $h = q/2$ ($h = -q/2$, respectively). In any case, the terminology is not to be confused with the usual notion of *chiral symmetry* for fields where holomorphic and antiholomorphic components decouple.

The chiral primary representations form an additive subgroup $\mathbb{Z}[\Delta_{\text{cp}}]$ of the ring of all NS representation classes. It is in general not invariant under products of NS representations, but truncating in a suitable manner the actual fusion rules (5.5), which are represented by certain coefficients $N_{ij}^k \in \mathbb{N}_0$, adjusts this point. Let \hat{N}_{ij}^k denote the Δ_{cp} -truncated fusion rules of the irreducible representations of \mathcal{A}_c^γ , namely

$$\hat{N}_{ij}^k := \begin{cases} N_{ij}^k : [i], [j], [k] \in \Delta_{\text{cp}} \\ 0 : \text{otherwise,} \end{cases} \quad (9.1)$$

and “ $*$ ” the corresponding product operation on $\mathbb{Z}[\Delta_{\text{cp}}]$.

Proposition 9.2. *The product $*$ defined by the Δ_{cp} -truncated fusion rules on $\mathbb{Z}[\Delta_{\text{cp}}]$ is commutative and associative. The corresponding ring is called the chiral ring.*

Proof. The chiral primary irreducible representations (l, m) of $\text{SVir}^{N=2}$ at given central charge c are characterized by the condition that $2h(l, m) = q(l, m)$. In light of Theorem 3.2(NS3), this means $m = -l$. Truncating then the fusion rules (5.5) according to this condition, at most one term remains in the sum (5.5), namely

$$(l_1, -l_1) * (l_2, -l_2) = \begin{cases} (l_1 + l_2, -l_1 - l_2) : l_1 + l_2 \leq n \\ 0 : \text{otherwise.} \end{cases} = (l_2, -l_2) * (l_1, -l_1),$$

for all $l_1, l_2 = 0, \dots, n$. In particular, this proves commutativity of $*$. Similarly, we check associativity:

$$\begin{aligned} ((l_1, -l_1) * (l_2, -l_2)) * (l_3, -l_3) &= \begin{cases} (l_1 + l_2, -l_1 - l_2) * (l_3, -l_3) : l_1 + l_2 \leq n \\ 0 : \text{otherwise.} \end{cases} \\ &= \begin{cases} (l_1 + l_2 + l_3, -l_1 - l_2 - l_3) : l_1 + l_2 + l_3 \leq n \\ 0 : \text{otherwise.} \end{cases} \\ &= (l_1, -l_1) * ((l_2, -l_2) * (l_3, -l_3)). \end{aligned}$$

□

Remark 9.3. In the physics literature the “chiral ring” often stands for the ring defined by the OPE at coinciding points of the the so-called chiral primary fields, cf. [40] and [1, Sec.5.3]. For the $N = 2$ minimal model considered here this gives the same result that we obtained by means of the chiral primary representations and their truncated fusion rules, see the example at the end of Section 5.5 in [1]. This is of course not surprising if one recalls that in the formulation of the approaches to CFT based on pointlike localized fields the fusion rules are defined in terms of the OPE of primary fields.

One can define in a similar way the rings corresponding to antichiral primary representations/fields. Note that the models we are considering are generated by fields depending on one light-ray coordinate only. In the general 2D case one has a richer structure of chiral/antichiral rings: (c, c) , (a, a) , (c, a) , (a, c) , cf. [40, pp. 433, 437].

Note also that from the OPE point of view the associativity of the ring product has a natural explanation, while it is not *a priori* evident from the point of view of the truncated fusion rules.

Recall from Section 4 that composition of a Neveu-Schwarz representation with the spectral flow endomorphism $\bar{\eta}_{1/2}$ gives a Ramond representation. Even more:

Proposition 9.4. *The spectral flow endomorphism $\bar{\eta}_{1/2}$ of \mathcal{A}_c gives rise to a one-to-one correspondence between Δ_R and Δ_{cp} .*

This is shown in a similar way as Proposition 5.6.

Remark 9.5. (1) It is natural to wonder whether the chiral ring has something to do with rings in noncommutative geometry, and whether one may obtain ring homomorphisms in this case. There is a natural ring structure in cyclic cohomology $HE^{2*}(\mathfrak{A}_{c,\Delta_R})$ coming from the cup product on the universal differential complex of \mathfrak{A} as discussed in [12, Sec.4.3.1]. According to Proposition 9.4 and the above discussion, we have a well-defined nontrivial semigroup homomorphism

$$\mathbb{Z}_+[\Delta_{cp}] \rightarrow HE^{2*}(\mathfrak{A}_{c,\Delta_R}), \quad \pi \mapsto \tau_{\bar{\eta}_{-1/2} \circ \pi},$$

from the chiral semiring (regarded as an additive semigroup) to the entire cyclic cohomology ring. It can, however, be shown by means of counterexamples that it does not preserve the ring structure. Therefore the answer to these questions seems negative so far.

(2) It is natural to look for the spectral-flow and the chiral ring (in the rational case) for general $N = 2$ superconformal nets, i.e., extensions of the super-Virasoro net \mathcal{A}_c . As a first step one can consider $N = 2$ superconformal nets with $c < 3$ which have been completely classified in Section 6. We hope to come back to these problems in the future.

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